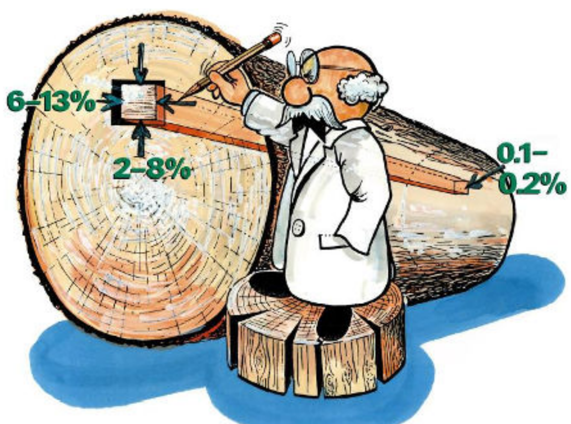
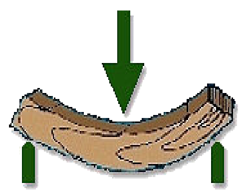
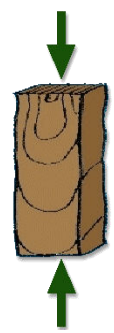
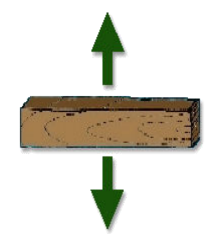
Lecture #1
Physical and Mechanical Properties of Wood

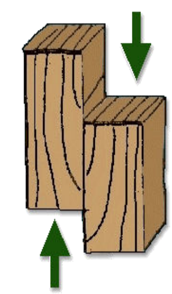
Y. H. Chui
University of Alberta





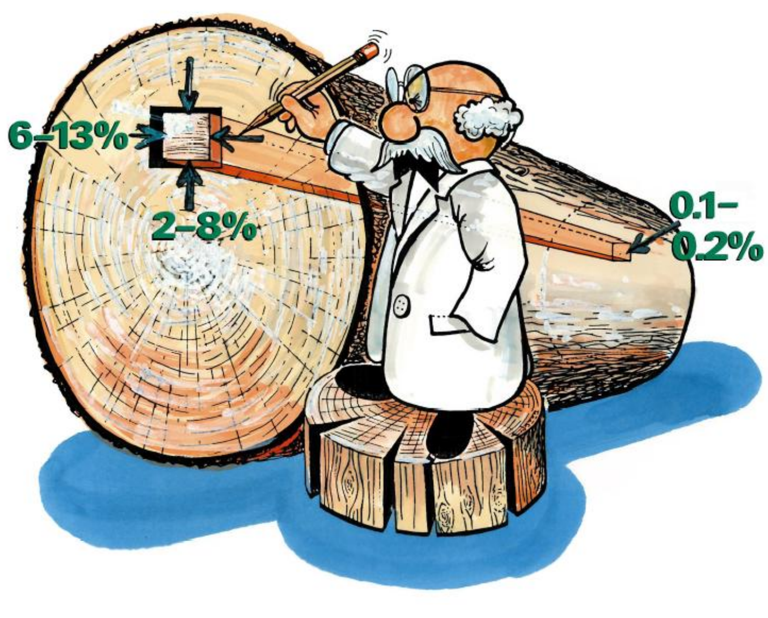






Part 1– Physical Properties of Wood

Typical Shrinkage Values for "Normal" Wood



Wood Density

- An important indicator of other wood properties:
  - Strength & stiffness
  - Thermal conductivity
  - Shrinkage/swelling
  - Paint adhesion
  - Glue adhesion
  - Drying

Definition of Density

$$\rho = \frac{m}{V}$$

- ρ: density
- m: mass
- V: volume of the piece of wood used in its determination

Units:

- pounds per cubic foot (lb/ft^3)
- grams per cubic centimetre (g/cm^3)
- kilograms per cubic metre (kg/m^3)
- $1\ lb/ft^3 \approx 16\ kg/m^3$

Wood Density

$$\rho_o = \frac{m_o}{V_o} = \frac{oven\text{-}dry\ mass}{oven\text{-}dry\ volume}$$

$$\rho_a = \frac{m_a}{V_a} = \frac{air\text{-}dry\ mass}{air\text{-}dry\ volume}$$

$$\rho_g = \frac{m_g}{V_g} = \frac{\text{green (saturated) mass}}{\text{green (saturated) volume}}$$

$$\rho_b = \frac{m_o}{V_g} = \frac{\text{oven - dry mass}}{\text{green (saturated) volume}}$$

Oven-dry, air-dry, green and bulk density.

Definition of Specific Gravity (Relative Density)

- In general terms, the ratio of the density (relative density) of a substance to that of water at the same temperature.
- In wood science,

$$SG = \frac{\text{Density of wood}}{\text{Density of water}} = \frac{OD\ mass / Volume\ at\ a\ MC}{Density\ of\ water}$$

- Specific gravity is always based on mass when oven-dry (OD), but volume can be at any given MC, e.g. green, oven dry or at any moisture content

Basic Specific Gravity

- Uses green volume basis. This is the most common value computed for wood, especially for research purposes.

$$SG_b = \frac{\frac{m_o}{V_g}}{\rho_{water}}$$

Oven-dry Specific Gravity

- Based on the oven-dry volume. It has application in computing the thermal diffusivity properties of wood and controlling quality of many wood products.

$$SG_o = \frac{\frac{m_o}{V_o}}{\rho_{water}}$$

Air Dry or Nominal Specific Gravity

- Based on air-dry (12% *MC*) volume. It is used as the basis for determining strength properties of wood.

$$SG_{12} = \frac{\frac{m_o}{V_{12}}}{\rho_{water}}$$

Basic SG of some hardwoods

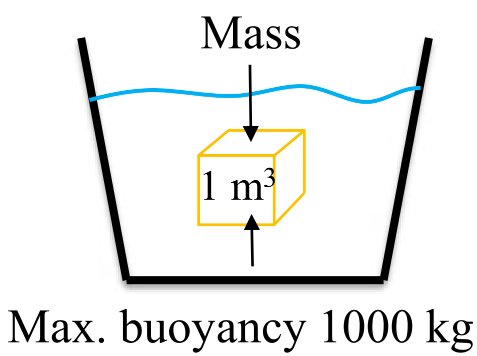
Species	Specific Gravity (12%)
white ash	0.63
aspen	0.41
basswood	0.38
yellow birch	0.66
hickory	0.77
sugar maple	0.66
red oak	0.65
white oak	0.72

Basic SG of some softwoods

Species	Specific Gravity (12%)
eastern white cedar	0.34
balsam fir	0.37
eastern hemlock	0.42
eastern white pine	0.37
jack pine	0.45
red pine	0.46
black spruce	0.43
red spruce	0.42
white spruce	0.37

Measurement of Specific Gravity

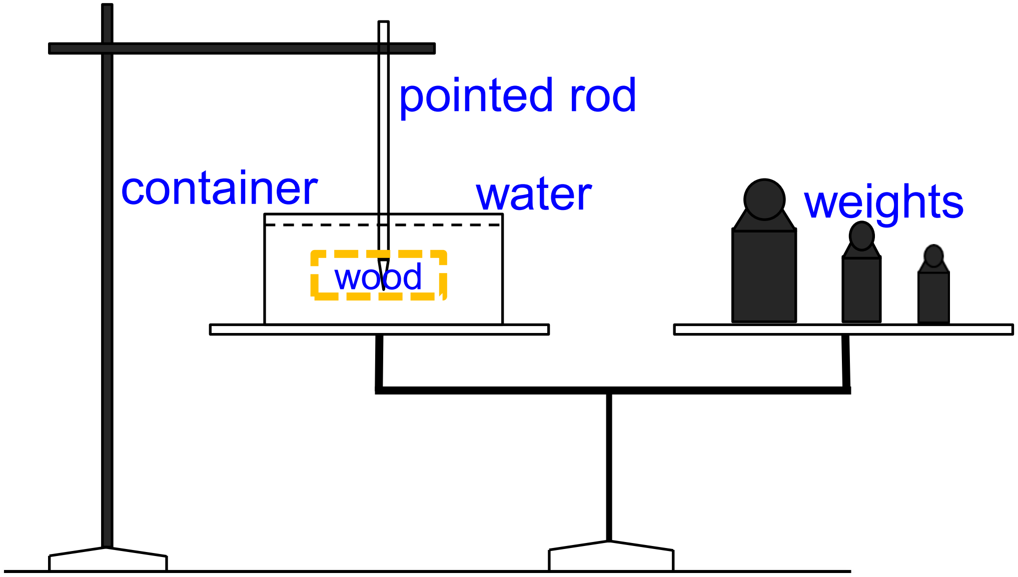
- ASTM D 2395–14 “Standard Test Methods for Specific Gravity of Wood and Wood-Based Materials”



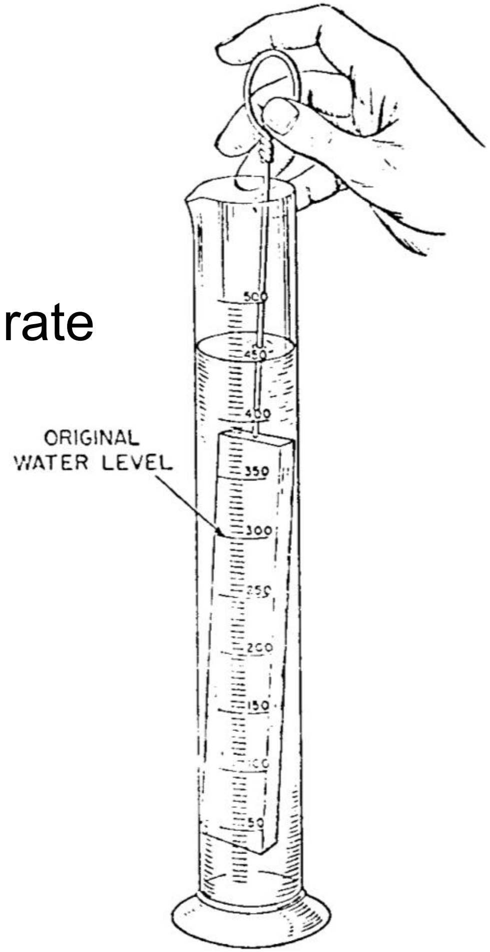
Density of water at 20° C=1000 kg/m^3

Measurement of dimensions

- Direct measurement of dimensions
  - For regular shape
- Immersion method
  - For irregular shape
  - Buoyancy weight of a specimen, more accurate
  - Dry specimen coated with wax



Diagrammatic sketch of apparatus used to measure volume of specimens



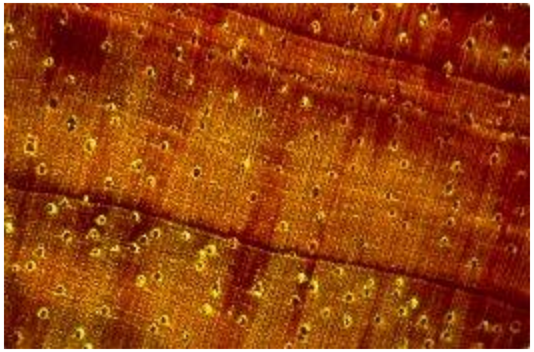
Measuring Volume of Elongated Specimens Using a Graduated Tube by Test Method B (Mode IV)

(Source: ASTM D 2395–14)

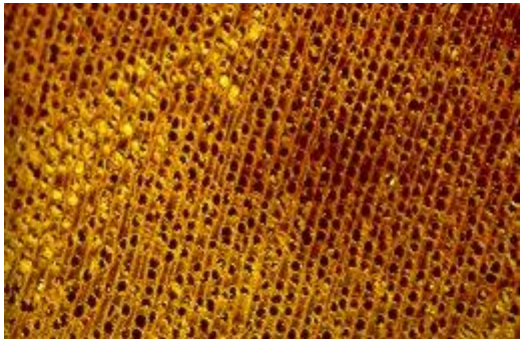
POROSITY / VOID VOLUME

Void spaces in wood include cell lumens, pit openings, pit cavities, and intercellular spaces.

$$Porosity = \left(1 - \frac{SG_{oven-dry}}{1.50}\right) \times 100$$



Cocobolo (South America)



Kapur (Southwest Asia)

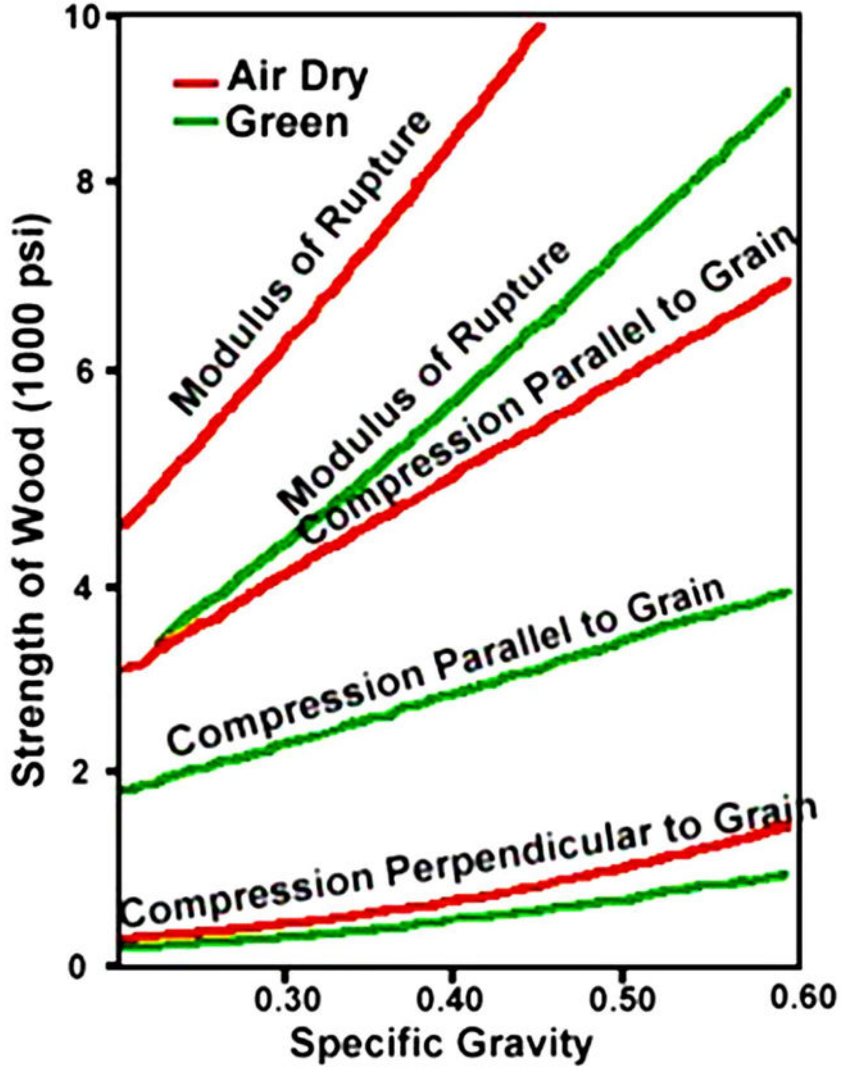
Estimation of *SG* at a given *MC*

Wood moisture content must be stated when referring to either density or specific gravity

$$SG_X = \frac{SG_{green}}{1 - S_V}$$

- SG_X: SG at X% moisture content
- SG_{green}: SG at green condition
- S_V: Volumetric shrinkage from green to X% moisture content

Effect of Specific Gravity on Mechanical Properties



WATER AND WOOD

- Wood is a hygroscopic material
- Physical and mechanical properties are greatly influenced by the amount of water present

Moisture Content

Moisture content (MC) is the mass of water contained in the wood, usually expressed as a percentage:

$$MC\ (\%) = \frac{mass\ of\ water\ in\ wood}{mass\ of\ ovendry\ wood} \times 100$$

$$MC(\%) = \frac{W_i - W_o}{W_0} \times 100 = \left(\frac{W_i}{W_0} - 1 \right) \times 100$$

$$where, W_i : initial\ mass\ of\ a\ wood\ block$$

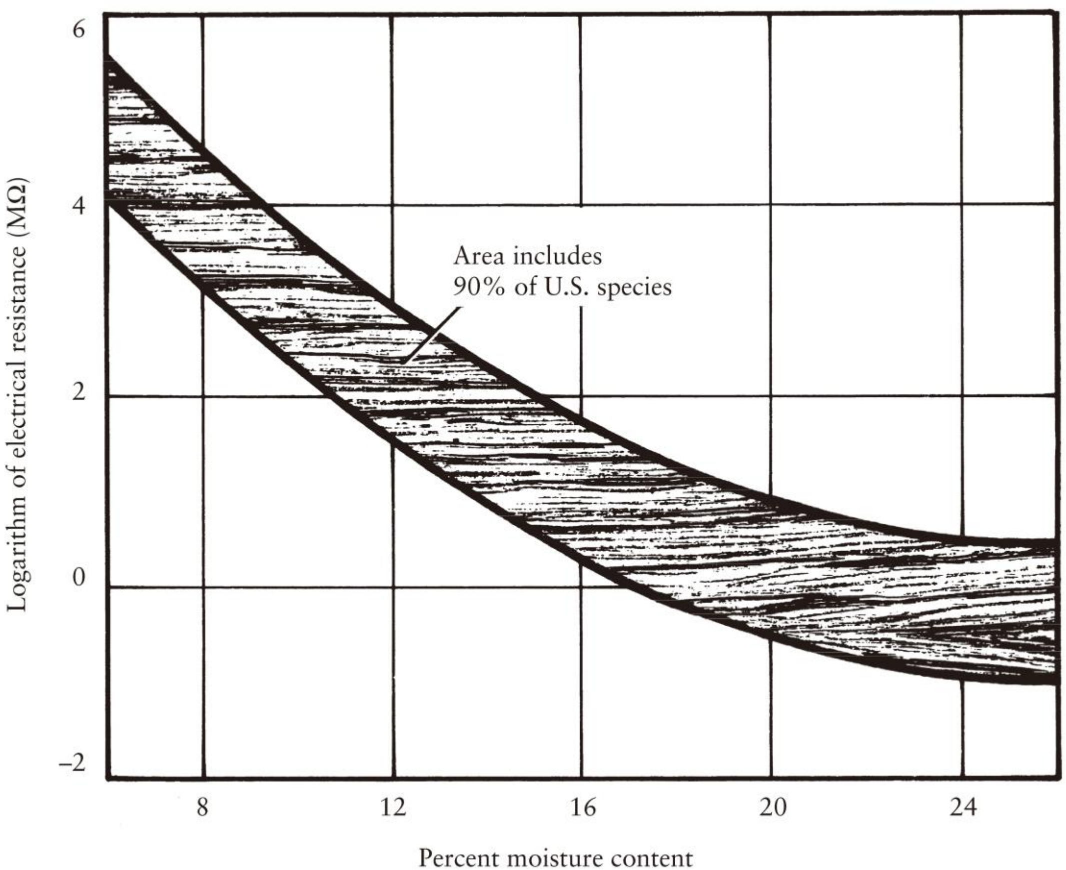
$$W_o : ovendried\ mass\ of\ the\ wood\ block$$

Measurement of Moisture Content

- Oven-dry method
  - Oven temperature: 103±2^{o}C
  - Accurate
- Electrical resistance method
  - Good for field measurement
  - Species and temperature dependent
  - Only works well between 6% and FSP (i.e. 28%)
  - Not very accurate (measurement error 3-5%)
- Distillation method
  - Measurement of moisture driven off
  - Good for wood with high extractive content
  - More accurate than oven-dry method
  - Delicate apparatus required

Electrical Resistance Method

Relationship of electrical resistance to moisture content



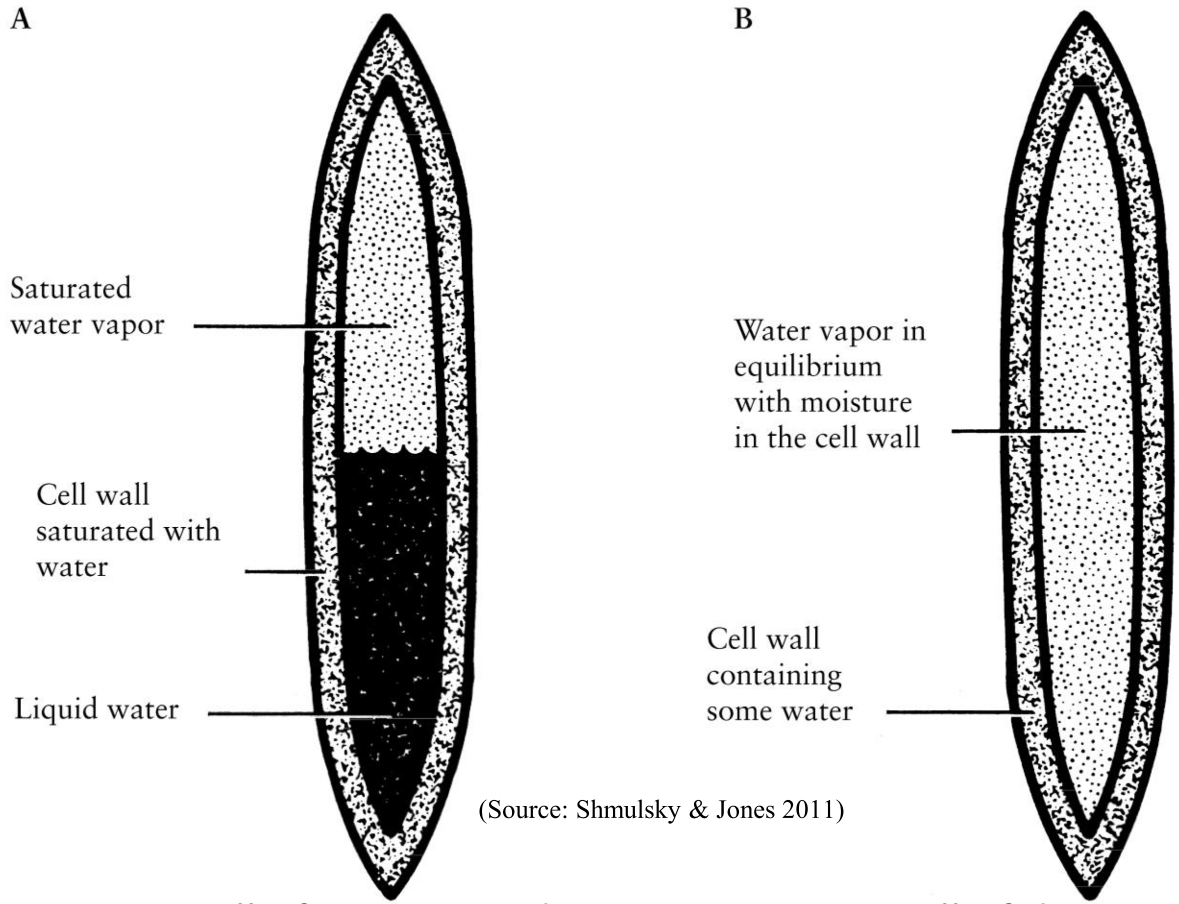
(Source: Glass and Zelinka 2010)

(Source: logismarket.de)

Measurement of Moisture Content

- Advanced production line methods
  - Infrared/radio frequency/microwave (suitable for high speed production)
  - Electrical capacitance

Location of Water in Wood



(Source: Shmulsky & Jones 2011)

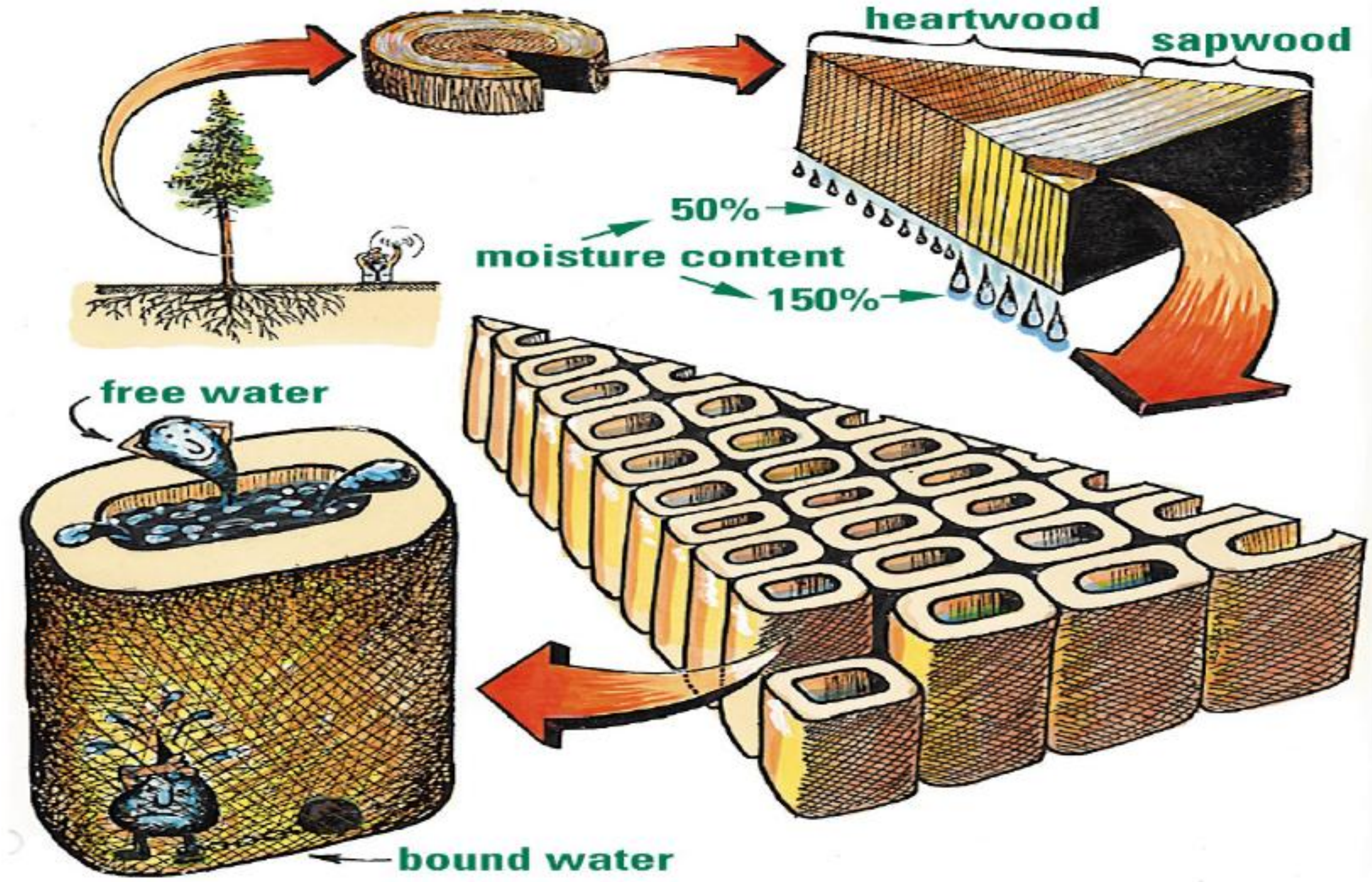
Water in a cell of green wood   Water in a cell of dry wood

- In voids such as cell lumens
- Within cell walls

Nature of Water in Wood

- Free water
  - in voids such as lumens, pit openings, pit cavities, and intercellular spaces
  - Influence weight only
- Bound water
  - held by physico-chemical forces (hydrogen bonds) in cell wall
  - Influence most physical and mechanical properties

Water in Softwood



Fibre Saturation Point (FSP)

- The point at which all the liquid water in the lumen has been removed but the cell wall is still saturated is called FSP.
- This is a critical point since below this most properties of wood are altered by changes in moisture content.
- MC at FSP is about 28%, ranging from 25% to 30%

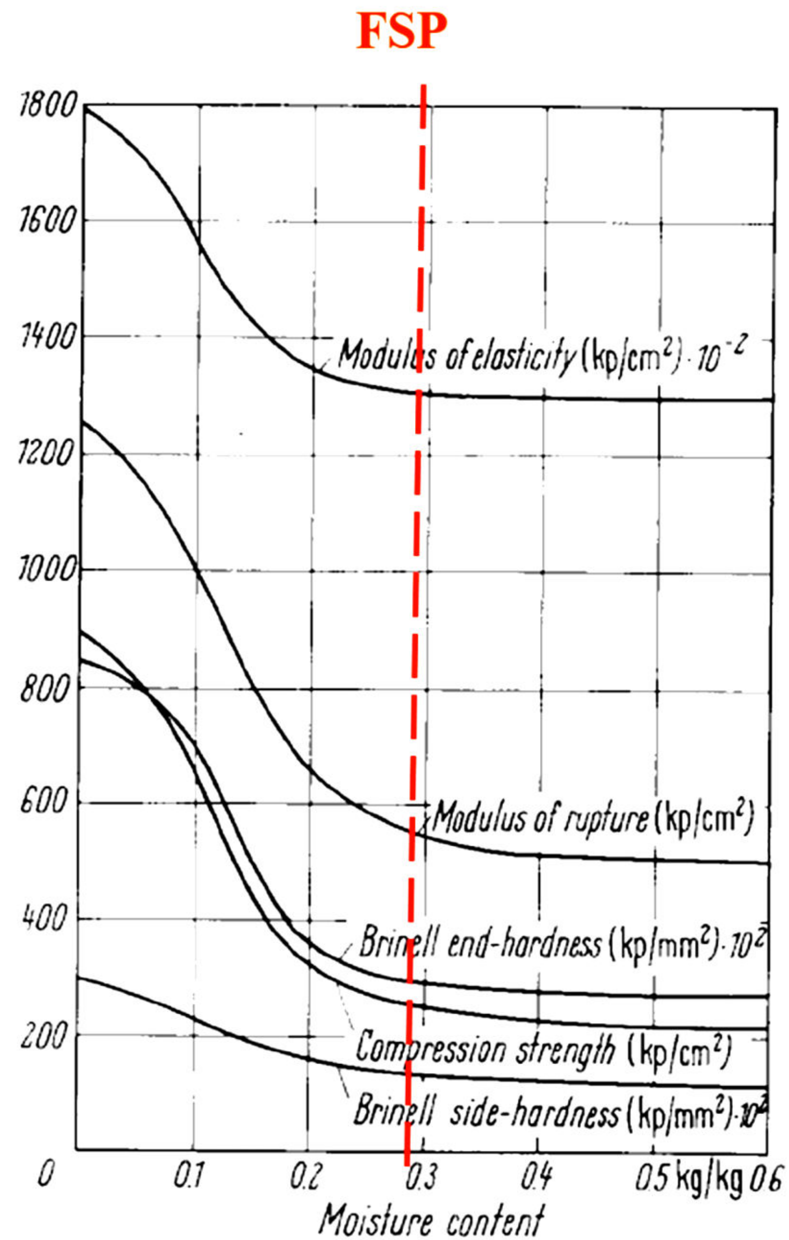


Fig. 6.48. Effect of moisture content in the hygroscopic range on mechanical properties for pine wood (*Pinus sylvestris* L.). From KOLLMANN (1944)

Factors influencing FSP

- Extractives such as resin, tannins, and fatty acids
- Chemical components
  - Cellulose: 40 - 50%
  - Hemicellulose: 20 - 30%
  - Lignin: 15 - 40%
- Temperature
- …

Fibre Saturation Point (FSP)

Fiber Saturation Point of Different Woods at Room Temperature

Species	Fiber Saturation Point, %	Species	Fiber Saturation Point, %
Ash, white	24.0	Pine, klinki	32.5
Basswood	32.0	loblolly	21.0
Birch, yellow	27.0	longleaf	25.5
Cedar, Alaska	28.5	red	24.0
western red	22.0	slash	29.0
Douglas-fir	26.0	shortleaf	30.0
Eucalyptus	30.0	Redwood	22.5
Fir, red	30.0	Spruce, Sitka	28.5
Hemlock, western	28.0	red	27.0
Larch, western	28.0	white	30.0
Mahogany	24.0	Tamarack	24.0
Oak, white	32.5	Teak	22.0
swamp	31.0	Yellow poplar	31.5

Equilibrium Moisture Content (EMC)

- Wood is a hygroscopic material, which constantly exchanges moisture with its surrounding atmosphere even in an equilibrium system. This depends on the surrounding atmosphere's moisture carrying capability that is proportional to its temperature and relative humidity.
- The MC at which water transfers in and out of wood equal to each other ie there is no net change in MC, is known as equilibrium moisture content (EMC).
- EMC is affected by temperature, RH and to a less degree on species.

Estimation of EMC for solid wood

$$EMC\ (\%) = \frac{1,800}{W}\left[\frac{Kh}{1-Kh} + \frac{K_1Kh + 2K_1K_2K^2h^2}{1 + K_1Kh + K_1K_2K^2h^2}\right]$$

where *h* is relative humidity (%/100, if 20% RH h=0.2)

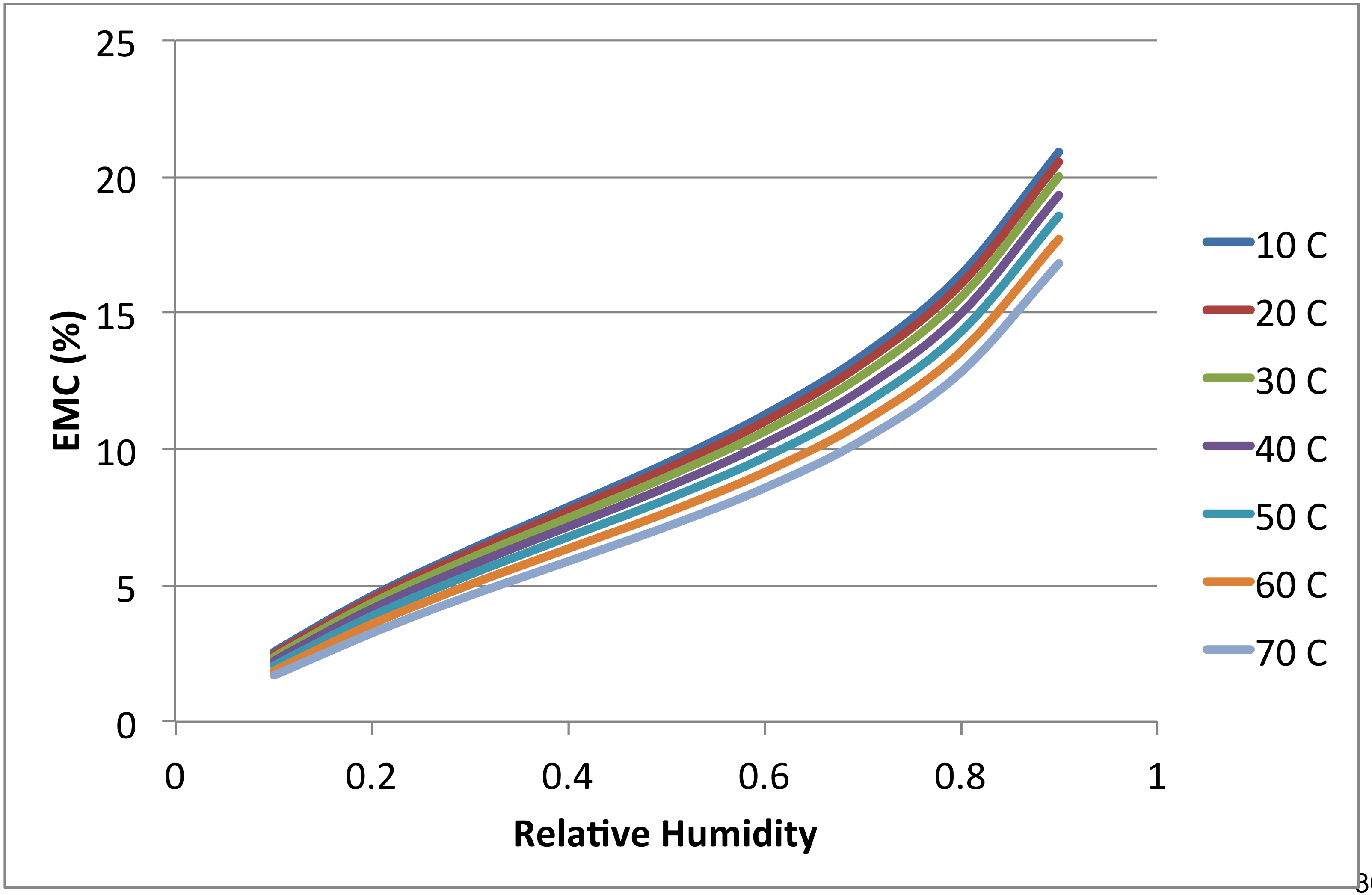
For temperature T in °C,

$$W = 349 + 1.29T + 0.0135T^2$$
$$K = 0.805 + 0.000736T - 0.00000273T^2$$
$$K_1 = 6.27 - 0.00938T - 0.000303T^2$$
$$K_2 = 1.91 + 0.0407T - 0.000293T^2$$

For temperature T in °F,

$$W = 330 + 0.452T + 0.00415T^2$$
$$K = 0.791 + 0.000463T - 0.000000844T^2$$
$$K_1 = 6.34 + 0.000775T - 0.0000935T^2$$
$$K_2 = 1.09 + 0.0284T - 0.0000904T^2$$

EMC vs RH (Wood)



EMC of Forest Products

TABLE 7.3. EMC of typical forest products at 21°C (70°F).

Relative humidity (%)	Moisture content (%) Wood	Softwood plywood	Particleboard	Oil-treated hardboard	High-pressure laminate
30	6.0	6.0	6.6	4.0	3.0
42	8.0	7.0	7.5	4.6	3.3
65	12.0	11.0	9.3	6.9	5.1
80	16.1	15.0	11.6	9.5	6.6
90	20.6	19.0	16.6	10.8	9.1

Source: Heebink (1966).

- EMC of wood products is slightly lower than the raw wood from which they are produced
  - Application of resins, coatings and sizing materials
  - Effect of heat treatment

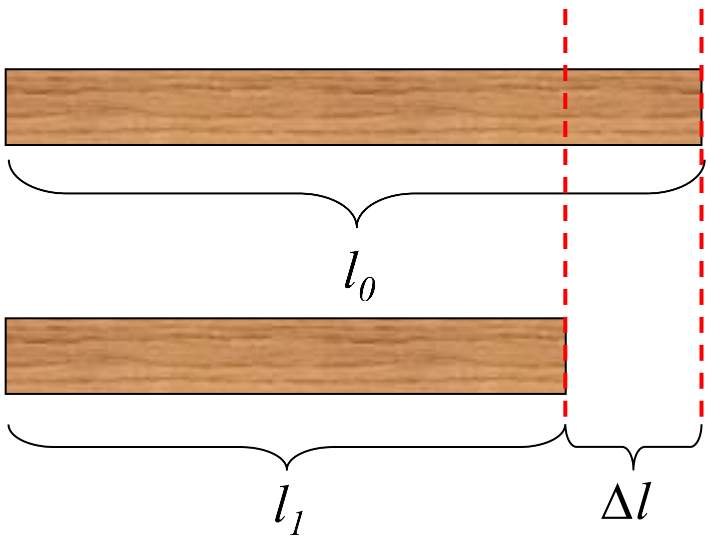
Shrinkage and Swelling

- Wood begins to shrink as it dries below the *FSP*, and continues to shrink until *MC* = 0.
- Wood swells as it absorbs moisture until *MC* = *FSP*. No further swelling occurs if *MC* > *FSP*.

Calculation of Shrinkage

<u>In wood science</u>

$$Shrinkage\ (\%) = \frac{decrease\ in\ dimension}{green\ (saturated)\ dimension} \times 100$$



$$Sh\ (\%) = \frac{\Delta l}{l_0} \times 100 = \frac{l_0 - l_1}{l_0} \times 100$$

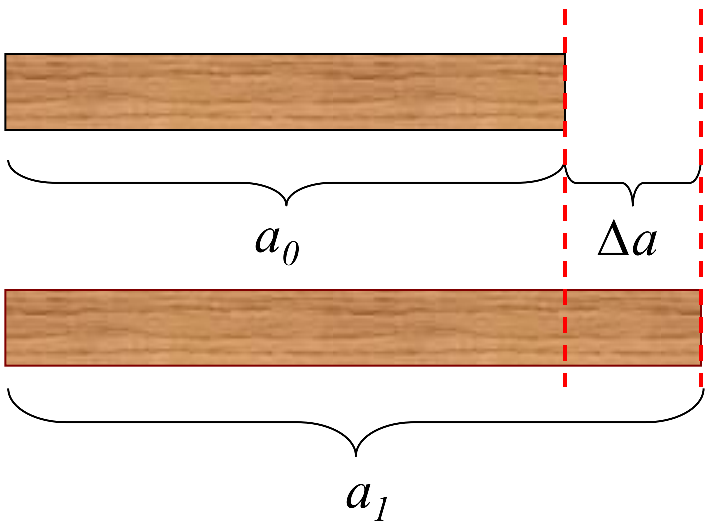
<u>Generally</u>

$$Shrinkage\ (\%) = \frac{Original\ dimension\ -\ Final\ dimension}{Original\ dimension} \times 100$$

Calculation of Swelling

<u>In wood science</u>

$$Swelling\ (\%) = \frac{increase\ in\ dimension}{ovendried\ dimension} \times 100$$

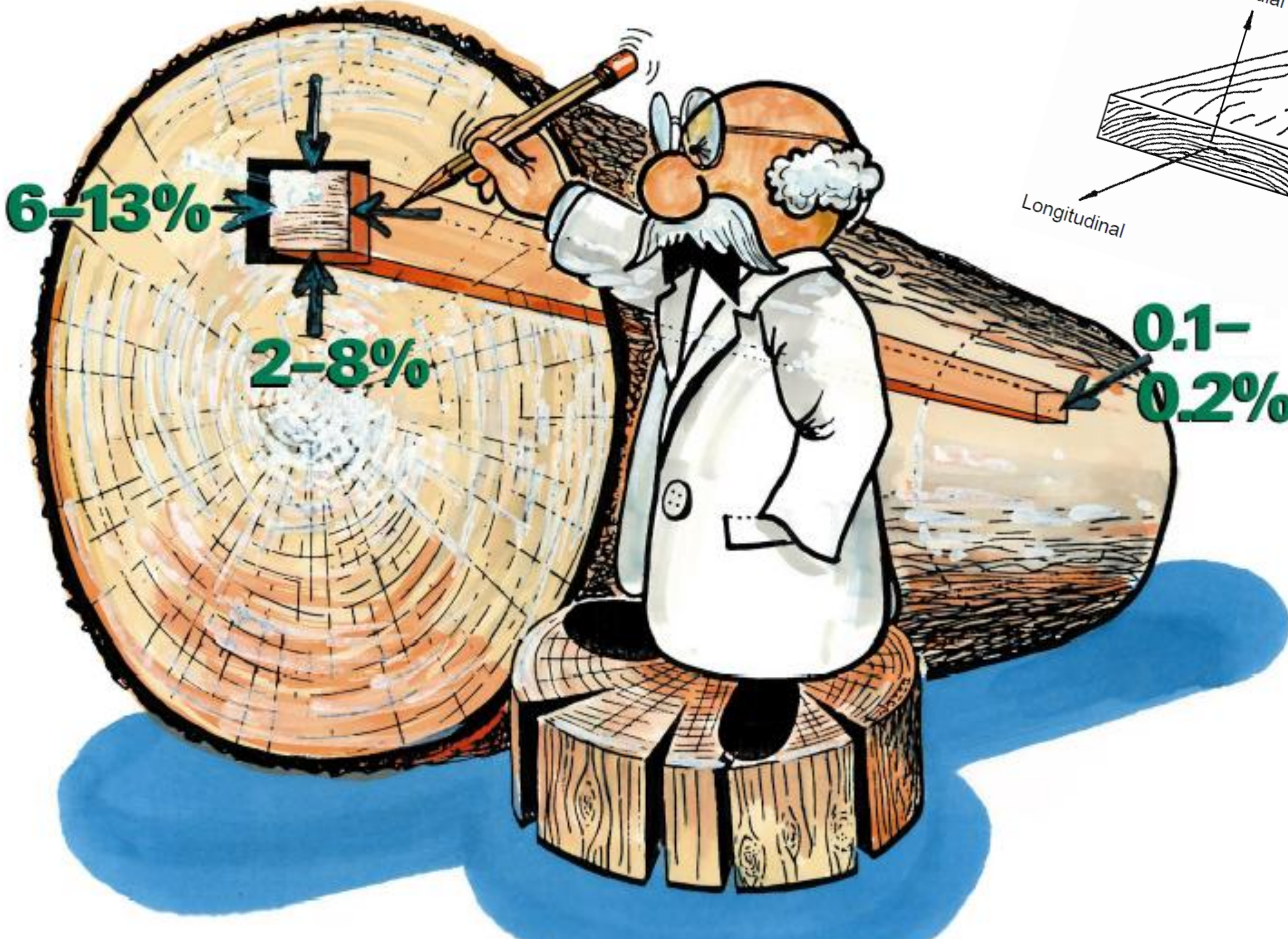


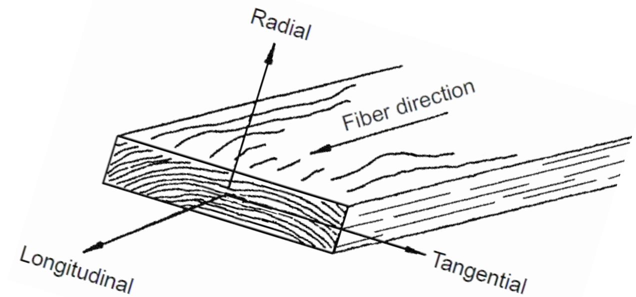
$$Sw\ (\%) = \frac{\Delta a}{a_0} \times 100 = \frac{a_1 - a_0}{a_0} \times 100$$

<u>Generally</u>

$$Swelling\ (\%) = \frac{Final\ dimension - Original\ dimension}{Original\ dimension} \times 100$$

Typical Shrinkage Values for "Normal" Wood





Anisotropy in Wood: direction-dependent. Generally assumed to be orthotropic in R, T and L directions)

Shrinkage values of wood from green to oven-dry moisture content

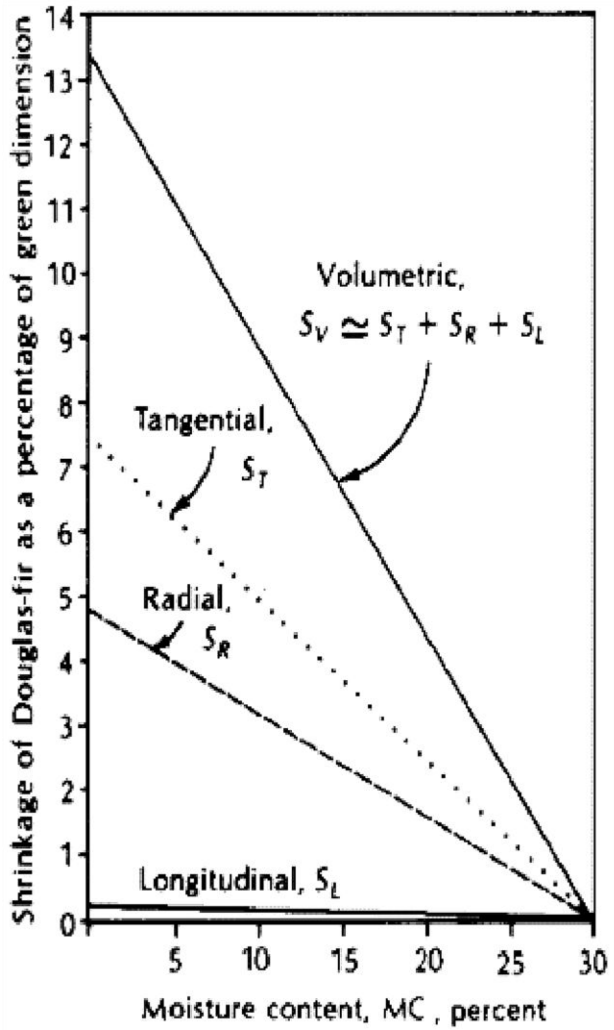
TABLE 7.5. Shrinkage values of wood from green to oven-dry MC.

Species	Shrinkage* (%)		
	Radial	Tangential	Volumetric
Domestic hardwoods			
White ash	4.9	7.8	13.3
Quaking aspen	3.5	6.7	11.5
Yellow birch	7.3	9.5	16.8
American elm	4.2	9.5	14.6
Sugar maple	4.8	9.9	14.7
Northern red oak	4.0	8.6	13.7
Black walnut	5.5	7.8	12.8
Imported hardwoods			
Apitong	5.2	10.9	—
Balsa	3.0	7.6	—
Mahogany	3.0	4.1	—
Teak	2.5	5.8	—
Khaya (African mahogany)	2.5	4.5	—
Softwoods			
Western red cedar	2.4	5.0	6.8
Coast Douglas fir	4.8	7.6	12.4
White fir	3.3	7.0	9.8
Western hemlock	4.2	7.8	12.4
Loblolly pine	4.8	7.4	12.3
Sitka spruce	4.3	7.5	11.5

*% Shrinkage = change in dimension/green dimension × 100.
Source: United States Forest Products Laboratory (USFPL) (1999).

Shrinkage difference in 3 directions

- Values of wood from FSP to oven-dry condition
  - Longitudinal shrinkage (S_L): 0.1-0.2%
    - Negligible for most practical purposes
  - Radial shrinkage (S_R): 2-8%
  - Tangential shrinkage (S_T) : 6-13%
  - Volumetric: $S_V \approx S_R + S_T$



Moisture content vs. shrinkage in Douglas-fir

Estimation of shrinkage at a given *MC*

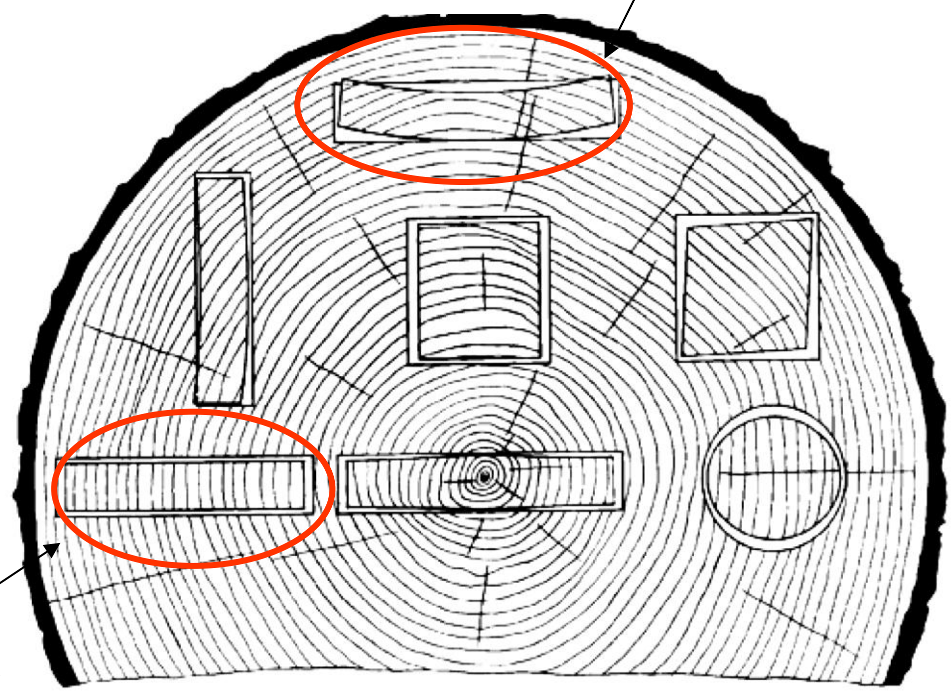
$$S_{MC} = S_O\left[\frac{FSP - MC}{FSP}\right]$$

- S_{MC} is the shrinkage from the green (*FSP*) condition to the *MC* of interest;
- S_O is the total shrinkage from the green to oven-dry condition.

Differential Shrinkage

- Reflection of anisotropic nature of wood
- Causing distortion in boards and lumber

A. Why does this piece warp this way after drying?



B. Why do the 4 sides of this piece remain square after drying?

Figure 4–3. Characteristic shrinkage and distortion of flat, square, and round pieces as affected by direction of growth rings. Tangential shrinkage is about twice as great as radial.

(Source: Wood Handbook 2010)

Warp

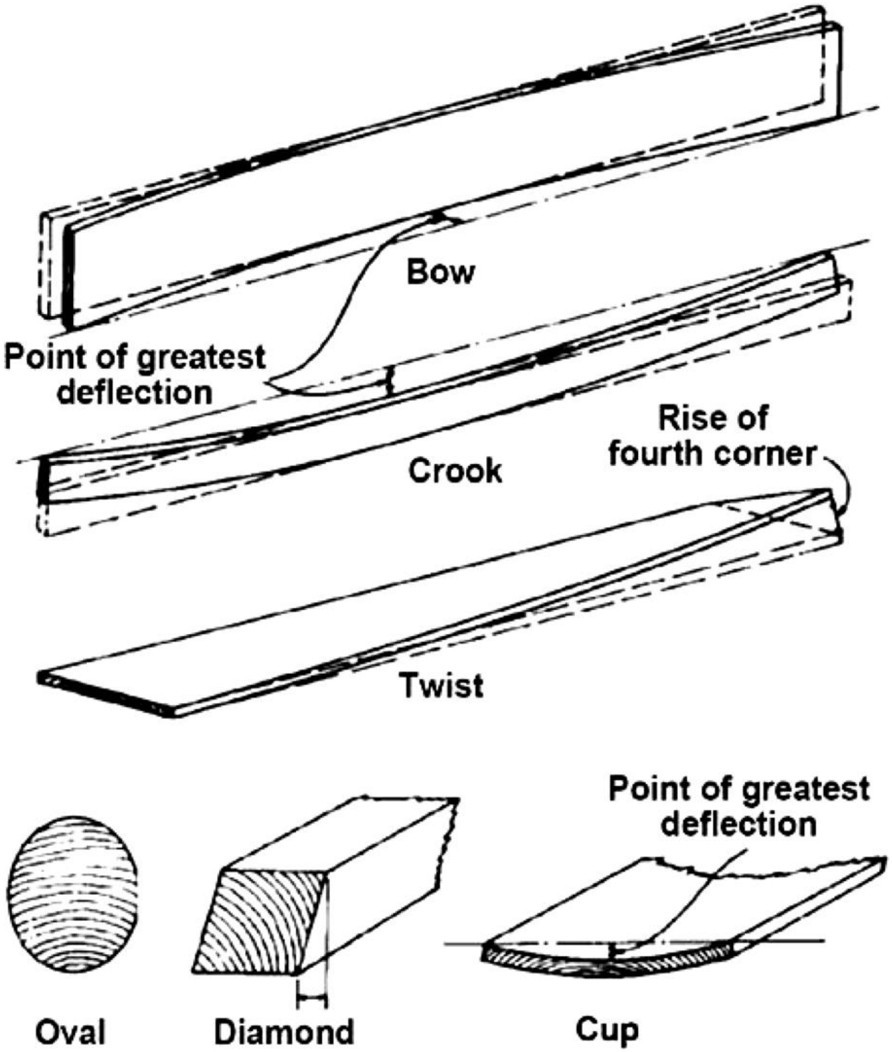


Figure 13–13. Various types of warp that can develop in boards during drying. (Source: Wood Handbook 2010)

Thermal Conductivity

- Thermal conductivity (K) is a measure of the rate of heat flow through a material due to a temperature gradient
- K parallel to grain is about 2 – 2.8 time that in perpendicular to grain direction
- K in transverse direction (perpendicular to grain) is related to SG and MC

$$K = SG_M (1.39 + C \cdot M) + 0.165, \quad \text{Btu/ (h. ft}^2 \cdot {}^\circ\text{F) per inch}$$

where $C = 0.028$ for $M < 40\%$

$= 0.038$ for $M \geq 40\%$

M = MC in %

SG_M = SG based on OD mass and volume at M

Thermal conductivity of selected species

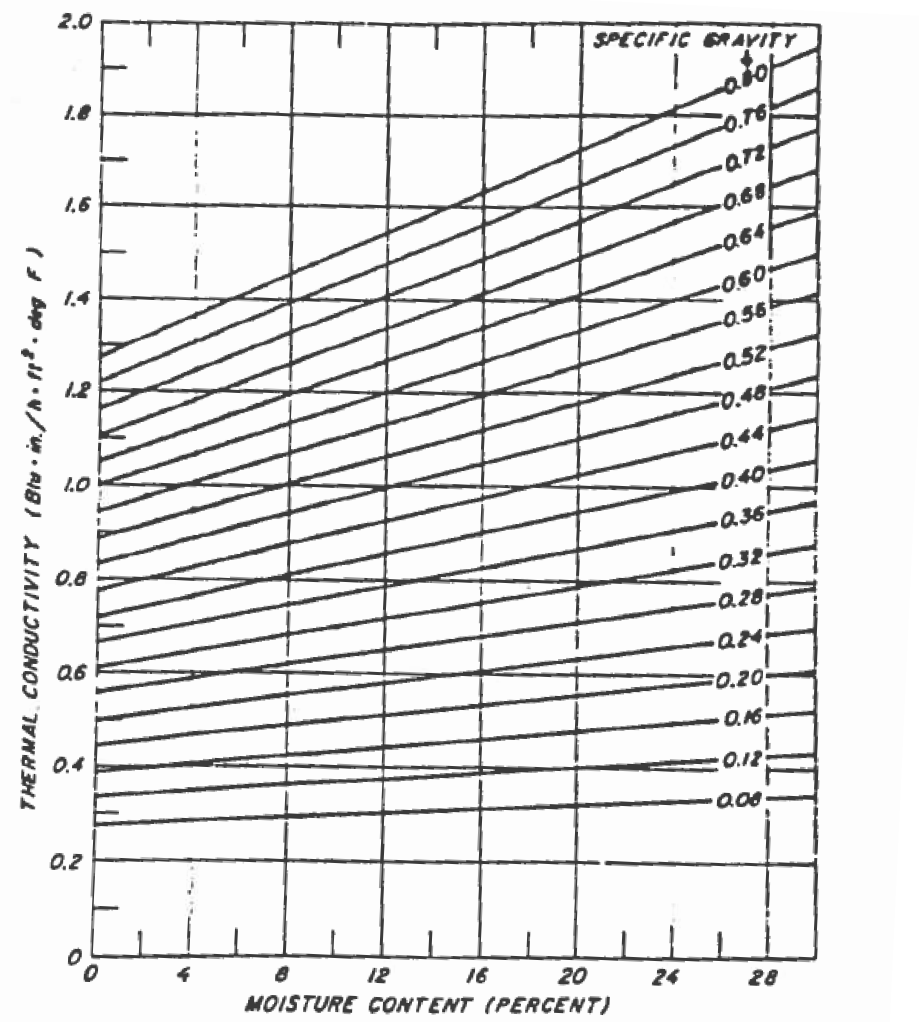
Table 4–7. Thermal conductivity of selected hardwoods and softwoods[a]—con.

Species	Specific gravity	Conductivity ($W\ m^{-1} K^{-1}$ ($Btu\ in.\ h^{-1} ft^{-2}\ °F^{-1}$))		Resistivity ($K\ m\ W^{-1}$ ($h\ ft^2\ °F\ Btu^{-1}\ in.^{-1}$))	
		Ovendry	12% MC	Ovendry	12% MC
Softwoods					
Baldcypress	0.47	0.11 (0.76)	0.13 (0.92)	9.1 (1.3)	7.5 (1.1)
Cedar					
Atlantic white	0.34	0.085 (0.59)	0.10 (0.70)	12 (1.7)	9.9 (1.4)
Eastern red	0.48	0.11 (0.77)	0.14 (0.94)	8.9 (1.3)	7.4 (1.1)
Northern white	0.31	0.079 (0.55)	0.094 (0.65)	13 (1.8)	11 (1.5)
Port-Orford	0.43	0.10 (0.71)	0.12 (0.85)	9.8 (1.4)	8.1 (1.2)
Western red	0.33	0.083 (0.57)	0.10 (0.68)	12 (1.7)	10 (1.5)
Yellow	0.46	0.11 (0.75)	0.13 (0.90)	9.3 (1.3)	7.7 (1.1)
Douglas-fir					
Coast	0.51	0.12 (0.82)	0.14 (0.99)	8.5 (1.2)	7.0 (1.0)
Interior north	0.50	0.12 (0.80)	0.14 (0.97)	8.6 (1.2)	7.1 (1.0)
Interior west	0.52	0.12 (0.83)	0.14 (1.0)	8.4 (1.2)	6.9 1.0)

(Source: Wood Handbook 2010)

Thermal Conductivity

- Metric units for K : Watt / m / Kelvin
- Conversion from Imperial to Metric – multiplied by 0.14422
- Thermal resistance, R = 1/K per inch



Thermal Expansion

- Thermal expansion

$$\alpha = \frac{\Delta \ell}{\ell}\left(\frac{1}{\Delta T}\right)$$

where $\Delta \ell$ = change in dimension

ΔT = change in temperature

ℓ = original dimension

- In R and T directions, α is dependent on SG_{OD}

$\alpha_R = (18 \times SG_{OD} + 5.5)\ 10^{-6}$ per ° F

$\alpha_T = (18 \times SG_{OD} + 10.2)\ 10^{-6}$ per ° F

- α is independent of SG in L direction

$\alpha_L = 1.7 – 2.5 \times 10^{-6}$ per ° F

Thermal Expansion

- To convert from Imperial to Metric units, multiplied by 1.8 i.e. per ° C
- Thermal expansion of other materials:

  Steel : 3 times

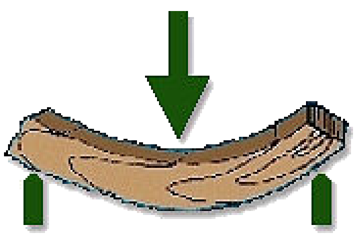
  Concrete : 2 times
- Thermal expansion of wood containing moisture is insignificant because it is offset by the shrinkage due to loss of moisture

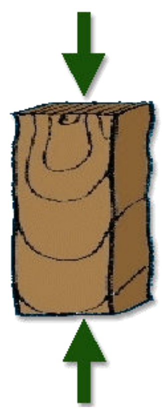
Combustion of wood

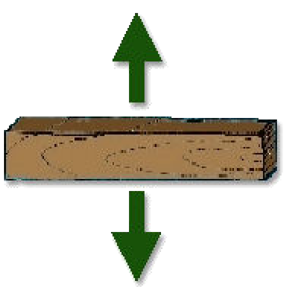
- Ignition temperature of wood is 275° C
- Speed of ignition depends on: 1) Size of wood; 2) Rate of heat loss from surface; 3) Rate of supply of heat to surface
- Heat of combustion (H) is defined as Btu produced by burning 1 pound of material
  - For OD wood, H = 8500 – 9000 Btu
  - Heat value is reduced by any presence of moisture in wood

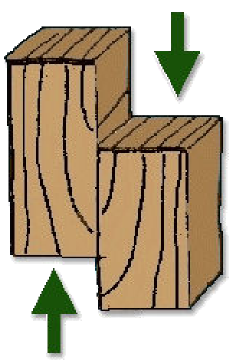
$$\text{Btu per pound of wood with moisture} = H \times \frac{100 - (M/7)}{100 + M}$$

Part 2– Mechanical Properties of Wood









Mechanical properties

- Behaviour of wood under different types of stress
- Methods of measurement for small clear wood
- Factors influencing mechanical properties of wood
  - Environmental (e.g. moisture, temperature)
  - Growth features (e.g. grain, knots, density)
  - Long-term loading
- Modification factors used in design

What are mechanical properties?

- Strength : the ability of a material to resist applied loads or forces without failure.
- Resistance to deformation (stiffness): change in dimension of a material when subjected to an applied load.

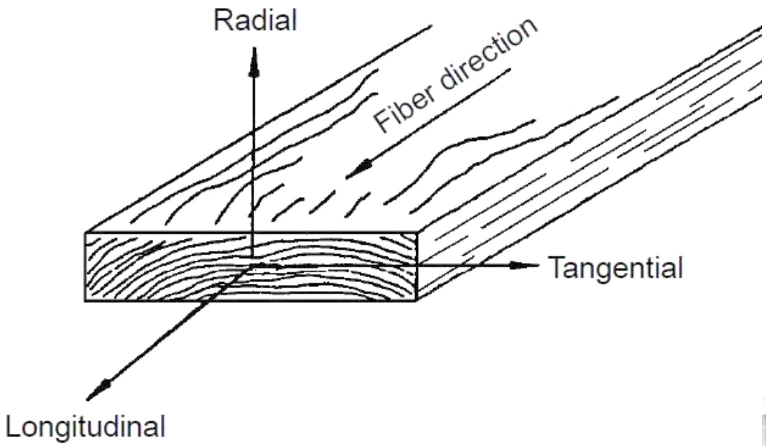
Comparison of wood with other materials

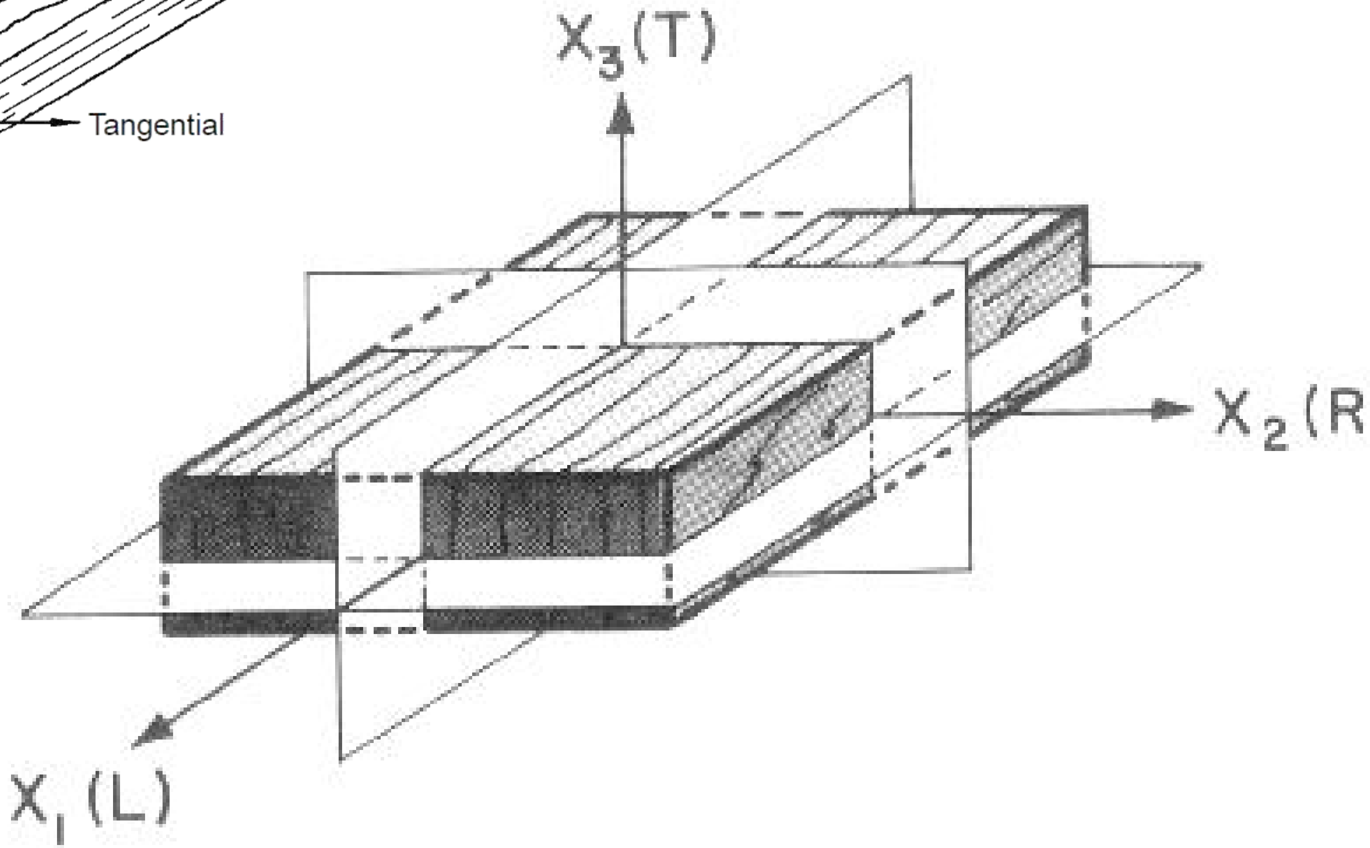
Material	Density (kg/m^3)	MOE (GPa)	Tensile strength (MPa)
Wood	500	12	80
Mild steel	7800	200	400
Aluminum	2700	70	200
Concrete	2300	35	3

Ratio of 'strength' to density

Material	MOE / density	Tensile strength / density	Compressive strength / density
Wood	20-30	120-170	60-90
Mild steel	26	30	30
Aluminum	25	180	130
Concrete	15	3	30

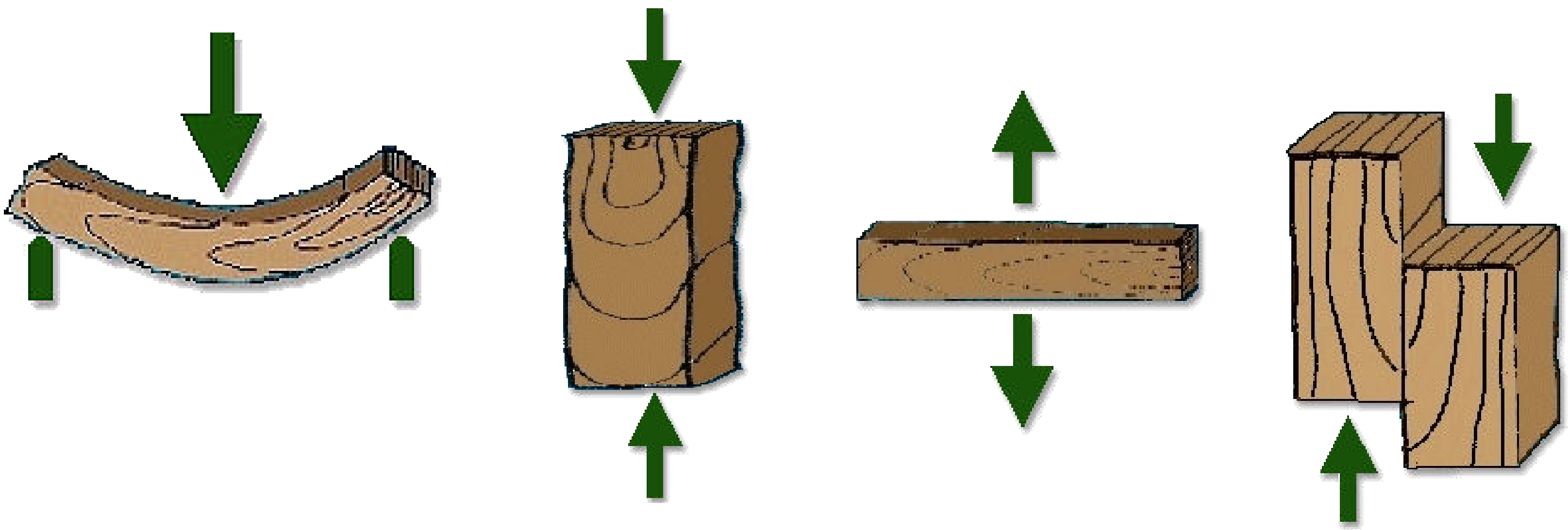
Orthotropic model of a clear wood block





(Source: Wood Handbook 2010)

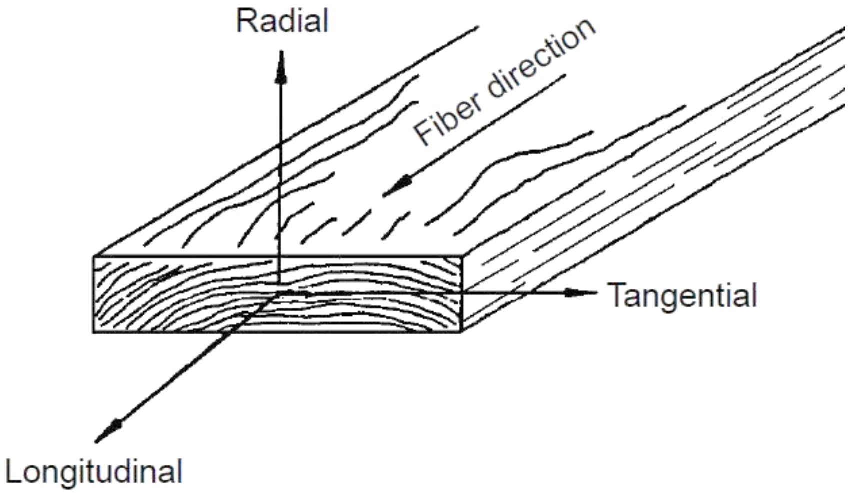
Types of loading



Bending Compression Tensile Shear

Mechanical Properties

- Orthotropic – properties being directionally dependent



$E_L : E_R : E_T \approx 20 : 1.6 : 1$

$G_{LR} : G_{LT} : G_{RT} \approx 10 : 9.4 : 1$

$E_L : G_{LR} \approx 14 : 1$

E – Young's modulus, G – shear modulus

- For practical purpose, assume two principal directions:
  - Parallel to grain (Longitudinal)
  - Perpendicular to grain (Tangential and Radial)

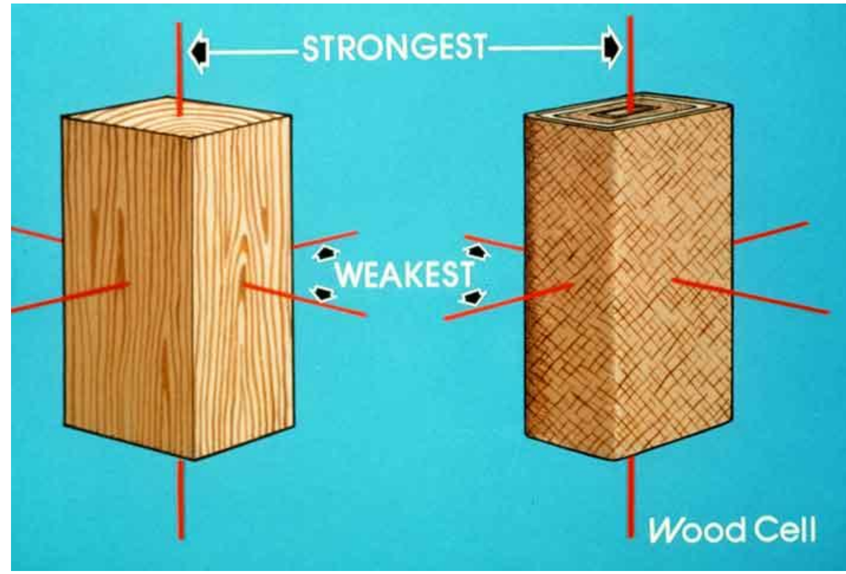
Mechanical Properties

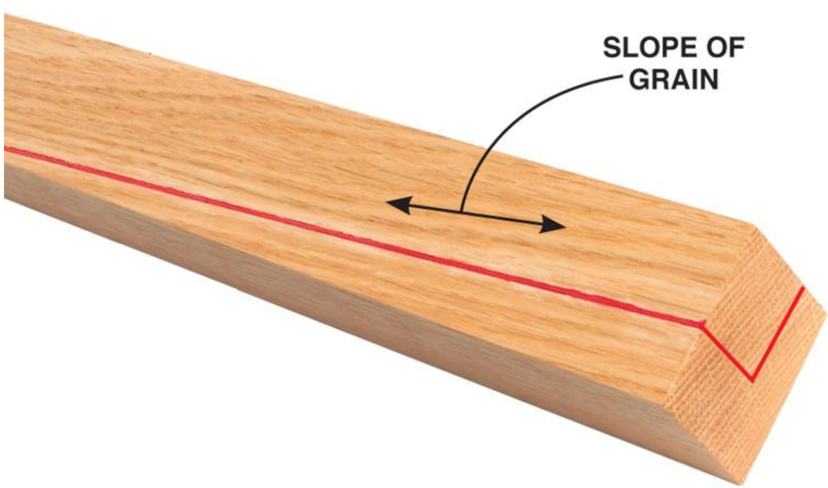
- Elastic properties
  - Modulus of elasticity (E): resistance to deformation caused by axial stress
  - Modulus of rigidity (G): resistance to deformation caused by shear stress
- Common strength properties
  - Modulus of rupture (MOR): strength of a member in bending (extreme fibre stress in tension or compression)
  - Compressive strength: parallel & perpendicular to grain
  - Tensile strength: parallel & perpendicular to grain
  - Shear strength: parallel to grain and through-thickness (rolling shear)

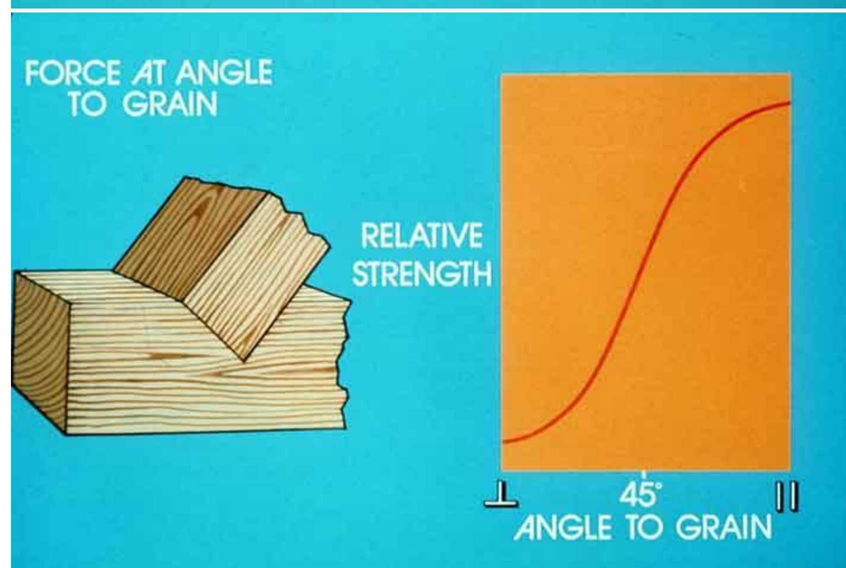
Strong in parallel-to-grain direction
Weak in perpendicular-to-grain direction

Strength of wood depends on loading direction relative to grain

- Parallel-to-grain strength ≈ 20 x perp-to-grain strength







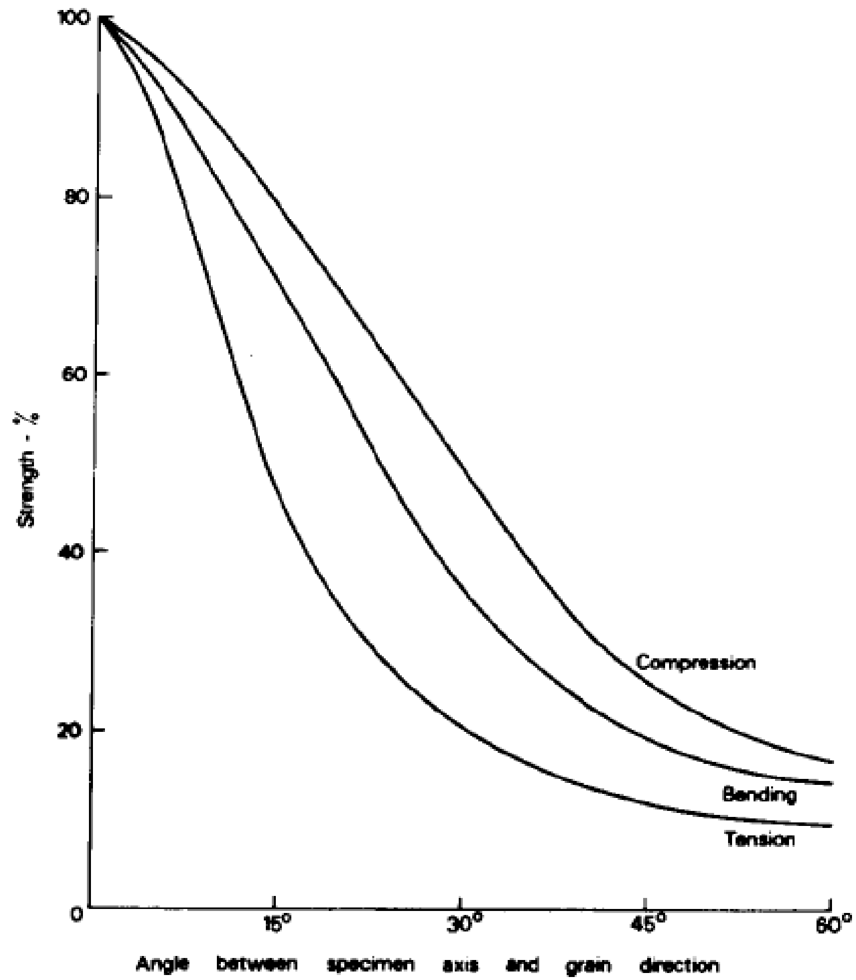
Hankinson's formula

Strength at angle θ, P_θ

$$P_\theta = \frac{P_{//} P_{\perp}}{P_{//} \sin^n \theta + P_{\perp} \cos^n \theta}$$

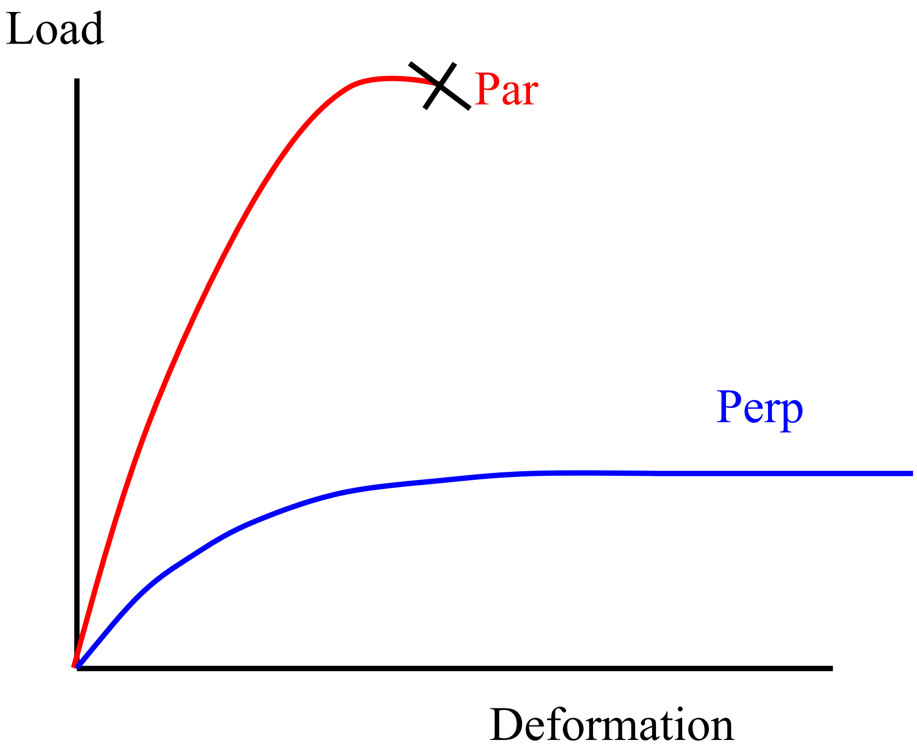
where $P_{//}$ and $P_{\perp}$ are strength parallel and perpendicular to grain respectively, n is dependent on property

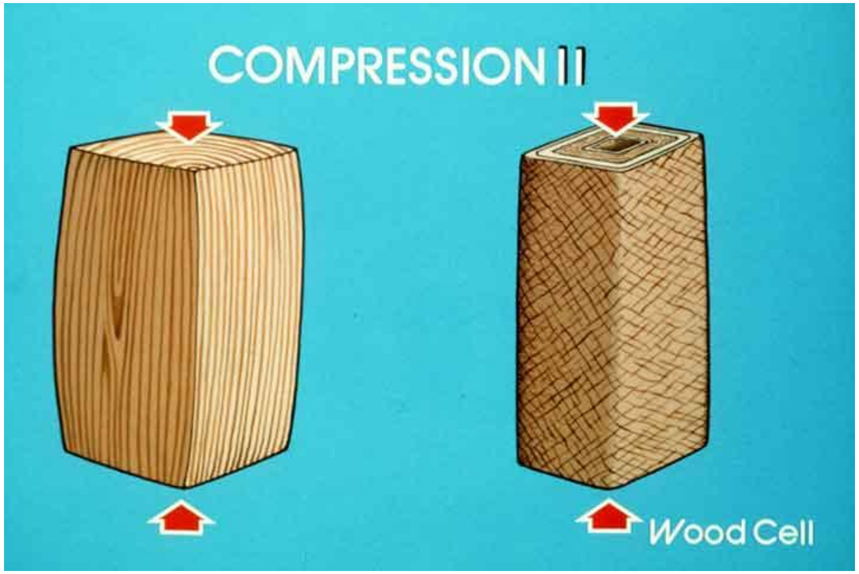
Property	*n*	$P_{\perp}/P_{//}$
Tension	1.5 - 2.0	0.04 - 0.07
Compression	2.0 - 2.5	0.03 - 0.40
MOR	1.5 - 2.0	0.04 - 0.10
MOE	2.0	0.04 - 0.12

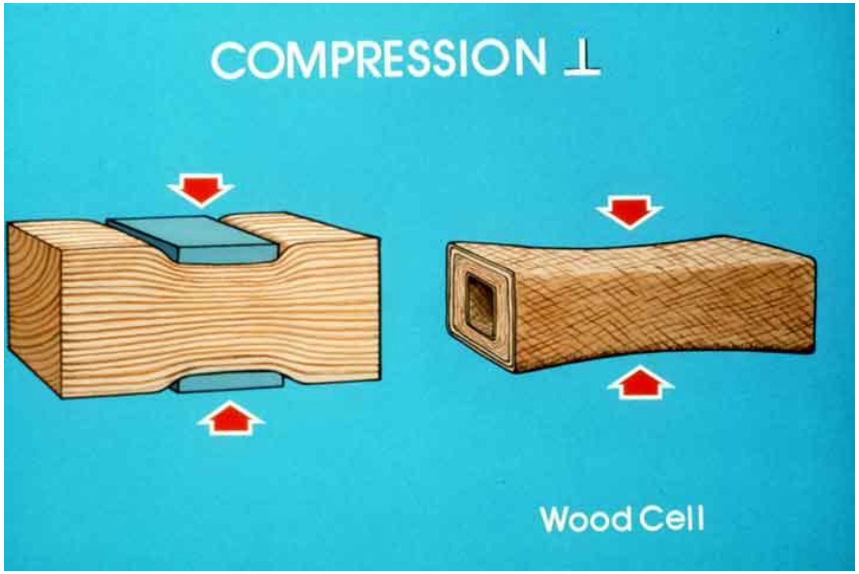


Compression

- Perp-to-grain is about 10%-20% that of par-to-grain strength

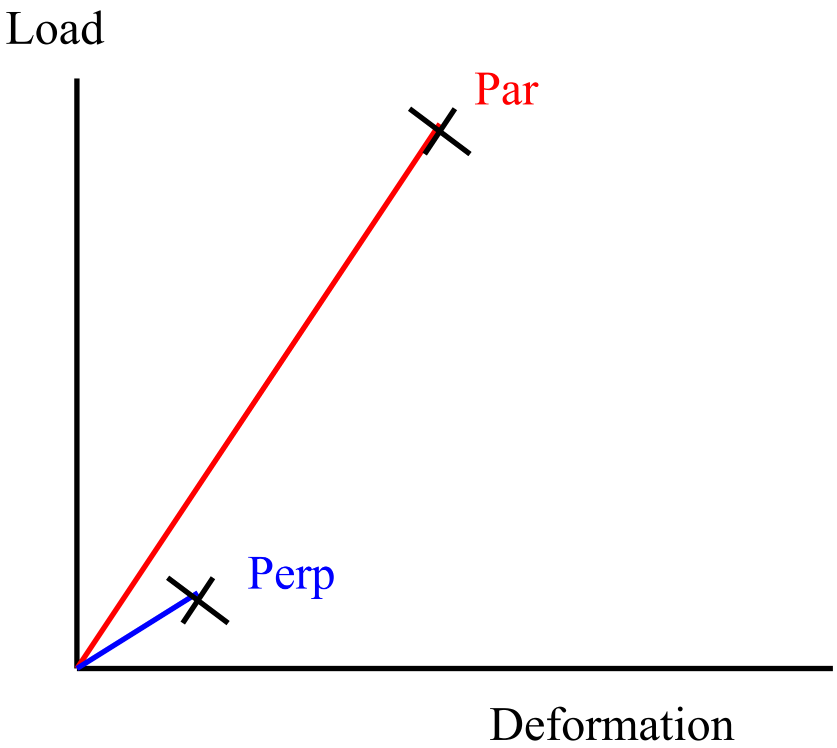


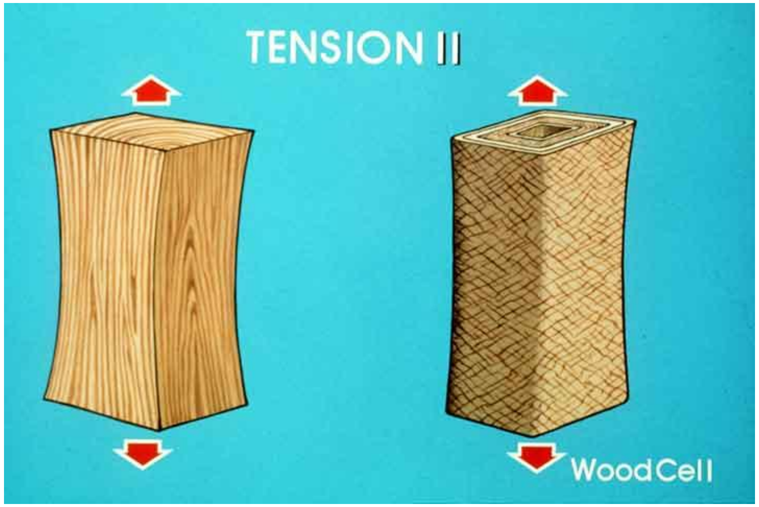




Tension

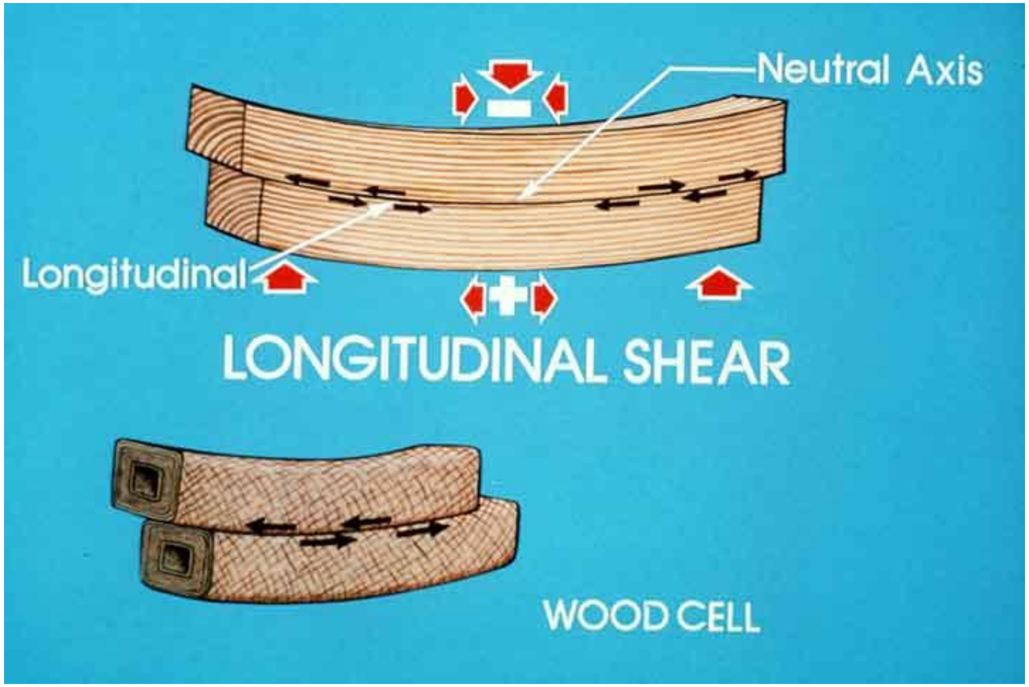
- Perp-to-grain strength is only about 5% that of par-to-grain





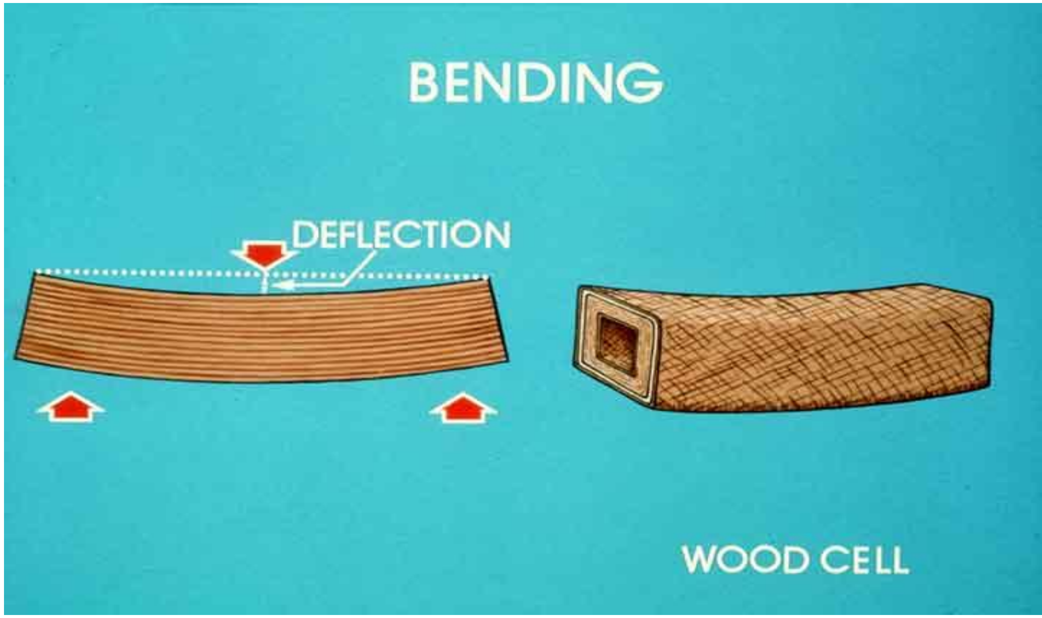
Shear parallel to grain (longitudinal shear)

- Not as affected by knot for tension and compression
- Property could be enhanced by knots
- Initial crack and splitting are weakness

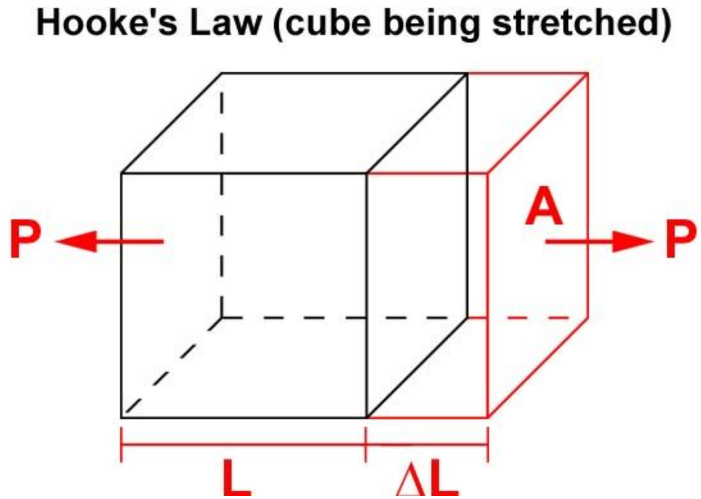


Bending

- Potential failure modes:
  - Tension
  - Compression
  - Shear
  - Bearing
- One of these would govern in practice
- Modulus of rupture (MOR) is often used for bending strength



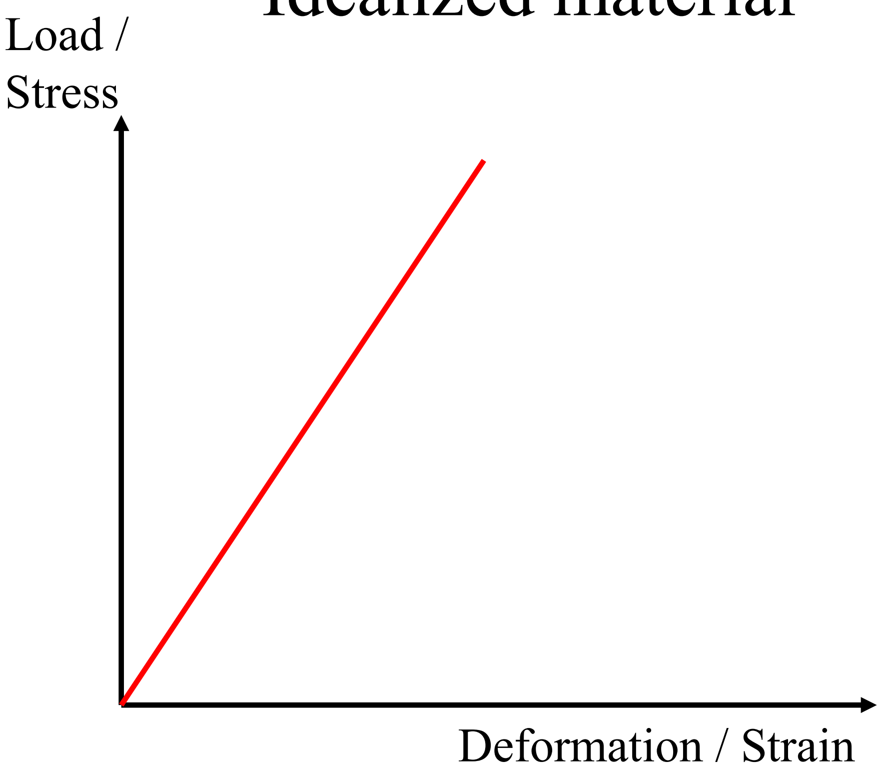
Hooke's law



$$Stress\ (\sigma) = \frac{Load\ (P)}{Area\ (A)}$$

$$Strain\ (\varepsilon) = \frac{Change\ in\ dimension\ (\Delta L)}{Original\ dimension\ (L)}$$

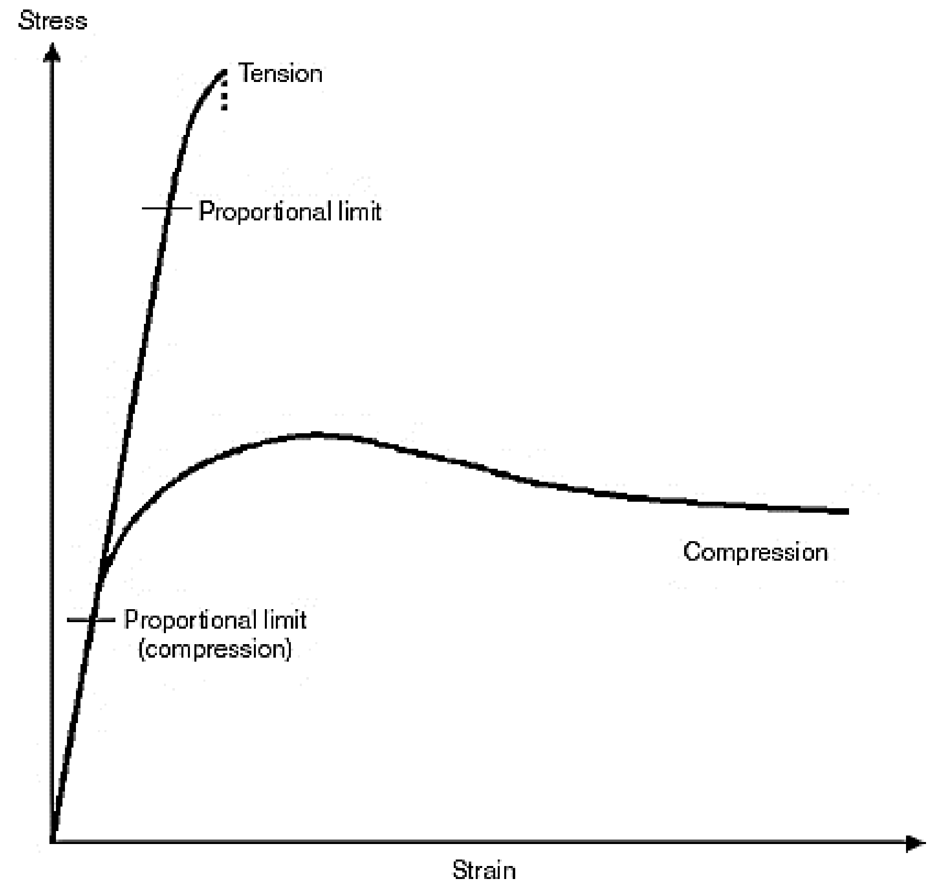
Idealized material



$$Young's\ modulus\ (E) = \frac{Stress\ (\sigma)}{Strain\ (\varepsilon)}$$

Stress-strain response

- Shape of stress-strain differs under different loading
- 3 key parts: Elastic, plastic and failure
  - Linear elasticity
  - Brittle and ductile failure
- Hooke's law
  - Proportional limit
  - Young's modulus (E) / Modulus of elasticity (MOE)



Stress versus strain curves for wood loaded in tension and compression

Relative strength among species

TABLE 9.4. Average strength values at 12 percent MC for defect-free straight-grained wood of selected species.

	Bending MOR		Bending MOE		Maximum crushing parallel to grain		Compression perpendicular to grain at proportional limit	
Species	Avg. kPa	Ratio*	Avg. MPa	Ratio	Avg. kPa	Ratio	Avg. kPa	Ratio
Hardwoods								
Quaking aspen	58,000	1.00	8,100	1.00	29,300	1.00	2,600	1.00
Red alder	68,000	1.17	9,500	1.17	40,100	1.37	3,000	1.19
Yellow poplar	70,000	1.20	10,900	1.34	38,200	1.30	3,400	1.35
Southern red oak	75,000	1.30	10,300	1.26	42,000	1.43	6,000	2.35
White ash	103,000	1.83	12,000	1.47	51,100	1.74	8,000	3.14
Sugar maple	109,000	1.88	12,600	1.55	54,000	1.84	10,100	3.67
Softwoods								
Eastern white pine	59,000	1.00	8,500	1.00	33,100	1.07	3,000	1.07
Engelmann spruce	64,000	1.08	8,900	1.05	30,900	1.00	2,800	1.00
Ponderosa pine	65,000	1.09	8,900	1.04	36,700	1.19	4,000	1.41
White fir	68,000	1.14	10,300	1.20	40,000	1.30	3,700	1.29
Radiata pine	80,700	1.37	10,200	1.20	41,900	1.36	4,200	1.50
Coastal Douglas fir	85,000	1.44	13,400	1.57	49,900	1.62	5,500	1.95
Scots pine	89,000	1.51	10,000	1.18	47,400	1.53	4,800	1.71
Longleaf pine	100,000	1.69	13,700	1.60	58,400	1.89	6,600	2.34

*Ratio of strength to weakest species in the group. (Source: Shmulsky & Jones 2011)

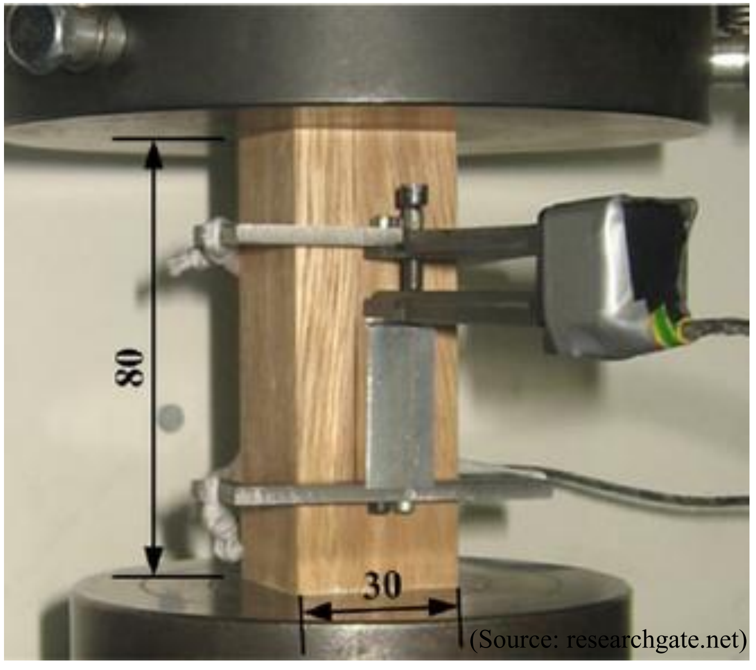
Mechanical properties

- Behaviour of wood under different types of stress
- Methods of measurement for small clear wood
- Factors influencing mechanical properties of wood
  - Environmental (e.g. moisture, temperature)
  - Growth features (e.g. grain, knots, density)
  - Long-term loading

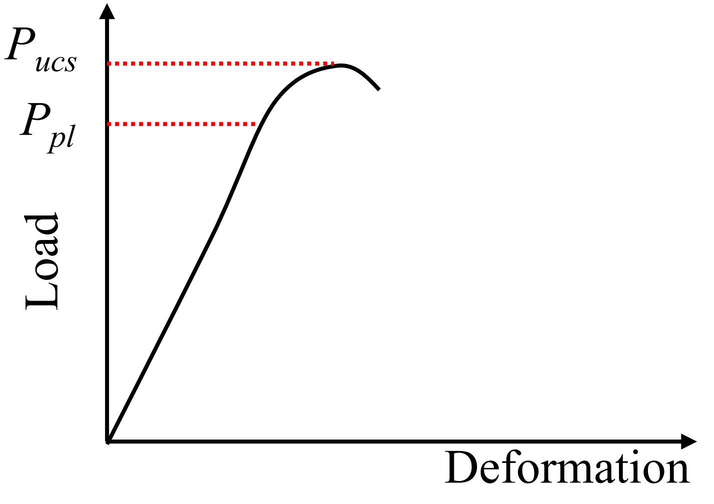
Test methods

- Strength properties of various species are determined through standardized tests of small, clear (defect free) specimens at various moisture conditions.
- Large pieces, containing defects typical of standard grades of lumber, are also tested to develop strength data (full-size test).

Compression parallel to grain

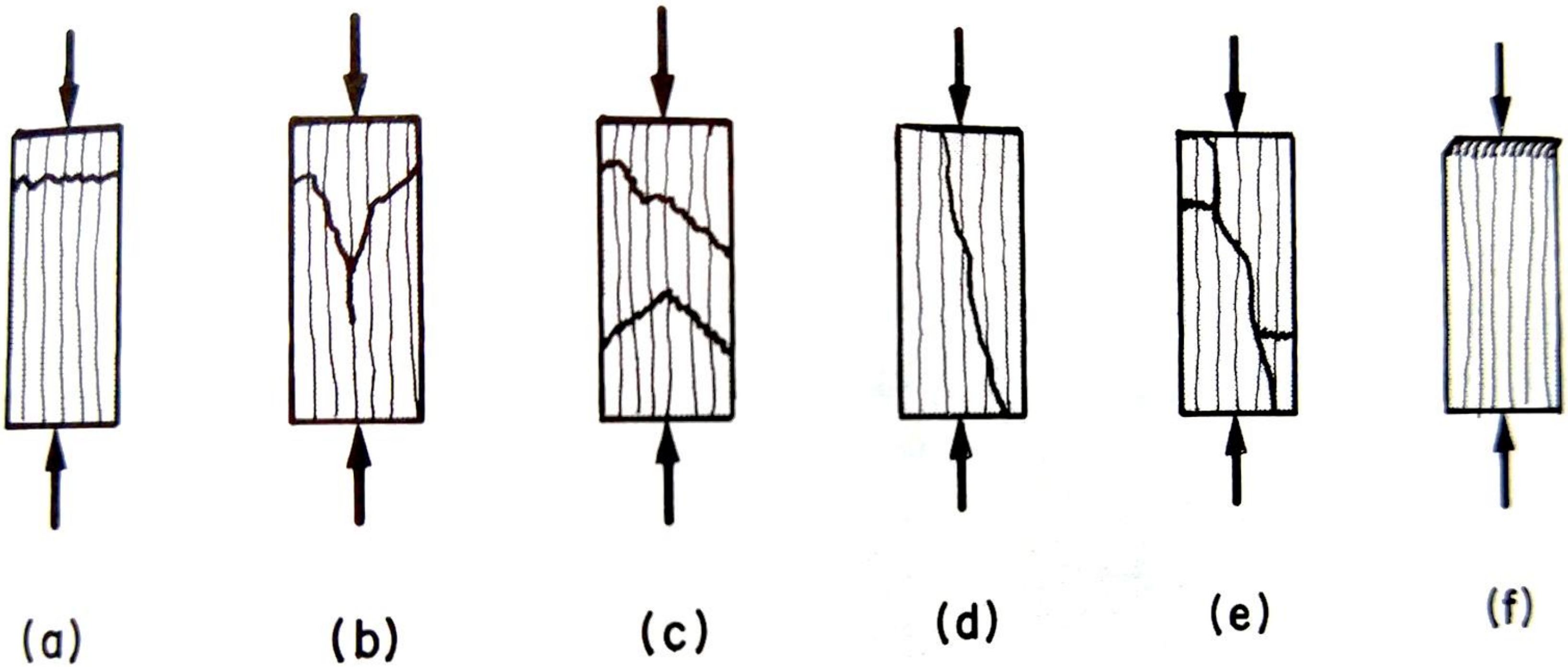


- Maximum compressive strength
- Modulus of elasticity
- Proportional limit



https://youtu.be/DFeHYFPElvE

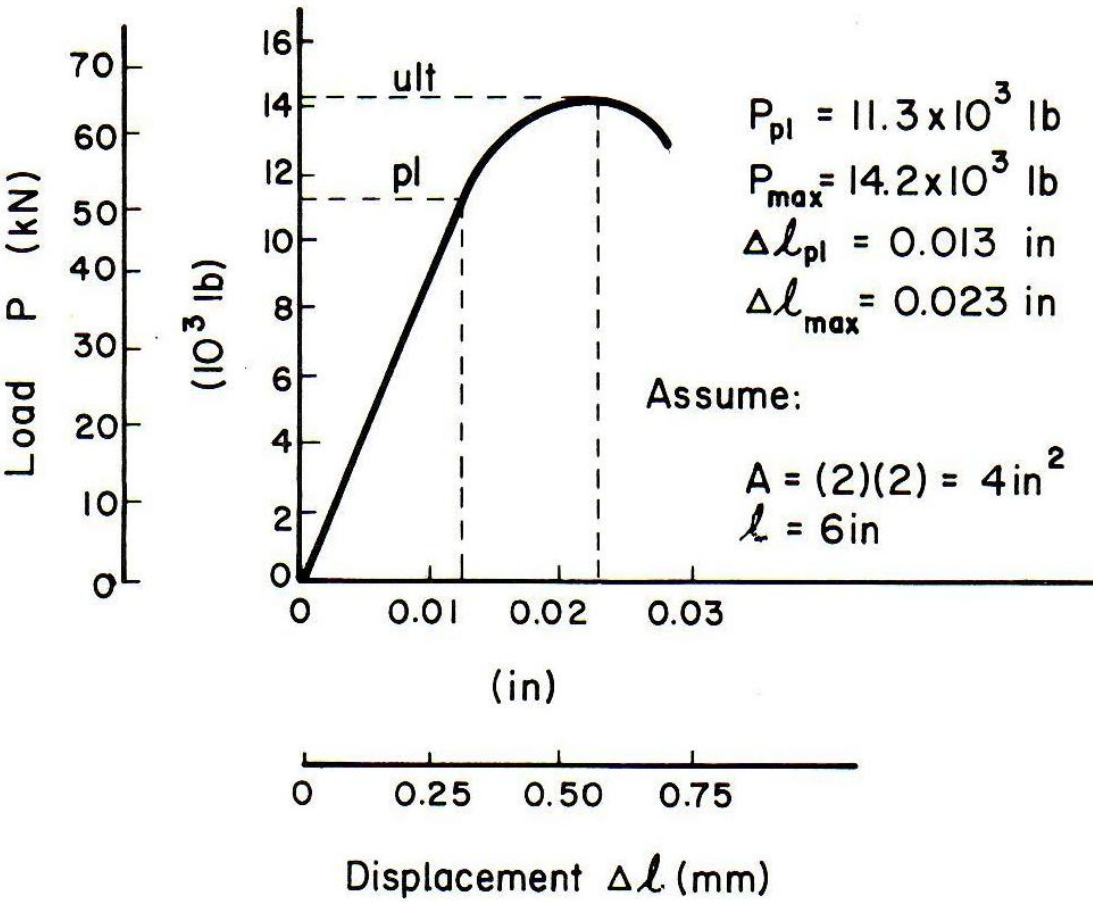
Failure modes



Failure types of nonbuckling clear wood in compression parallel to grain: (a) crushing, (b) wedge splitting, (c) shearing, (e) crushing and splitting, (f) brooming and end rolling.

(Source: Bodig & Jayne)

Example: compression parallel to grain



1) Proportional limit

$$\sigma_{pl} = \frac{P_{pl}}{A} = \frac{11300}{4}$$
$$= 2825\, psi\,(19.48\, MPa)$$

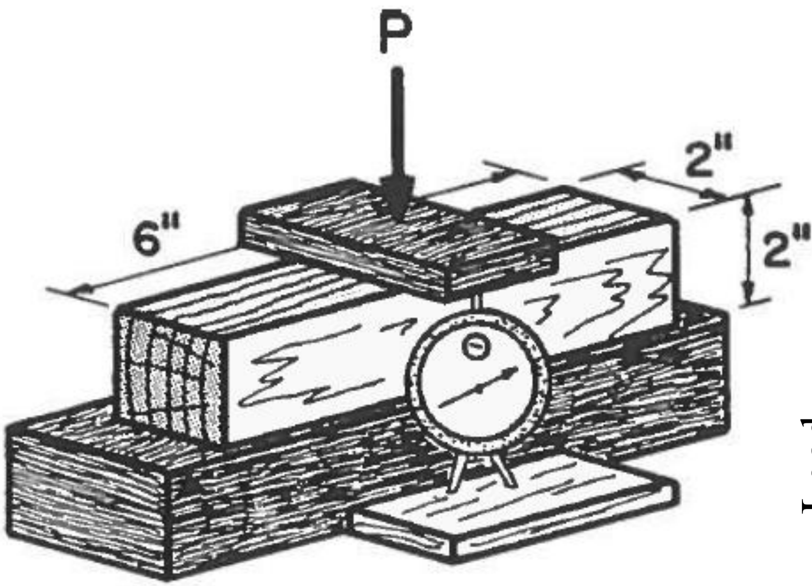
2) Modulus of elasticity

$$MOE = \frac{\sigma_{pl}}{\varepsilon_{pl}} = \frac{\frac{P_{pl}}{A}}{\frac{\Delta l_{pl}}{l}} = \frac{\frac{11300}{4}}{\frac{0.013}{6}} = \frac{2825}{0.00217}$$
$$= 1300\times 10^6\, psi\,(8.963\, GPa)$$

3) Maximum compressive strength

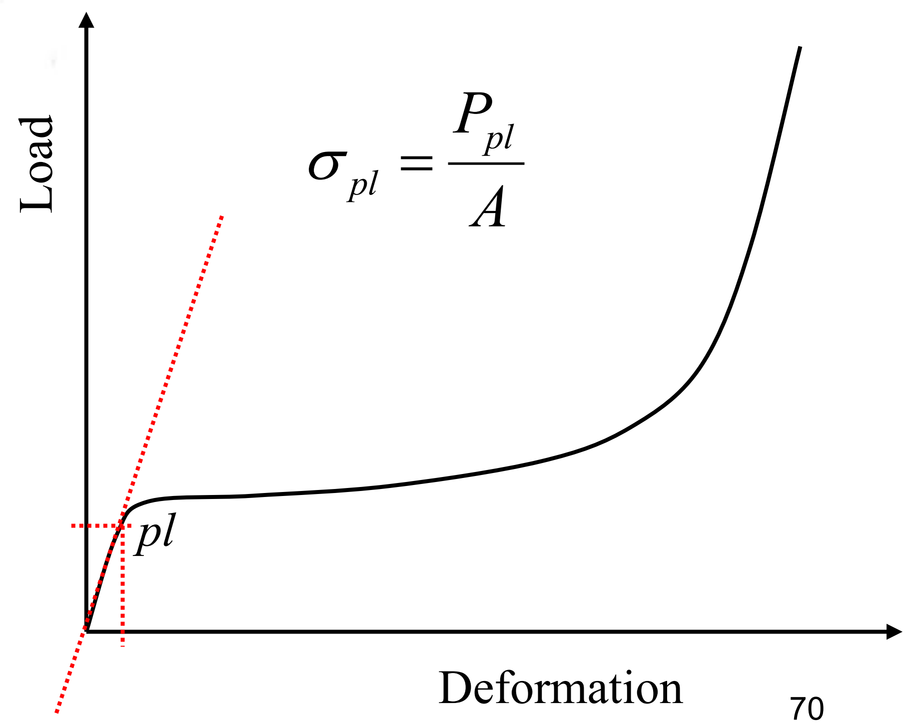
$$\sigma_{max} = \frac{P_{max}}{A} = \frac{14200}{4}$$
$$= 3550\, psi\,(24.48\, MPa)$$

Compression perpendicular to grain



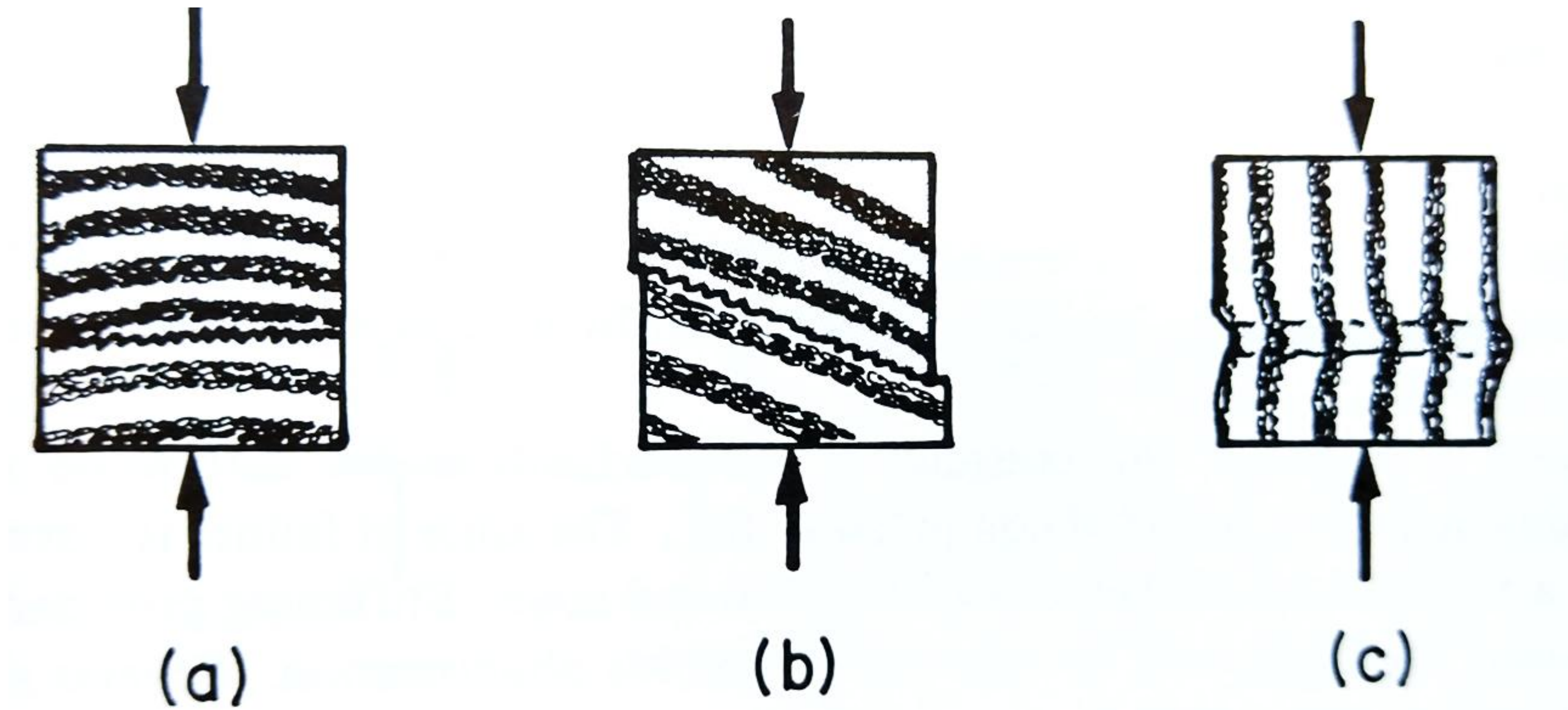
https://youtu.be/yqAFSKlALwk

- Proportional limit
- Modulus of elasticity
- May not have a clear failure



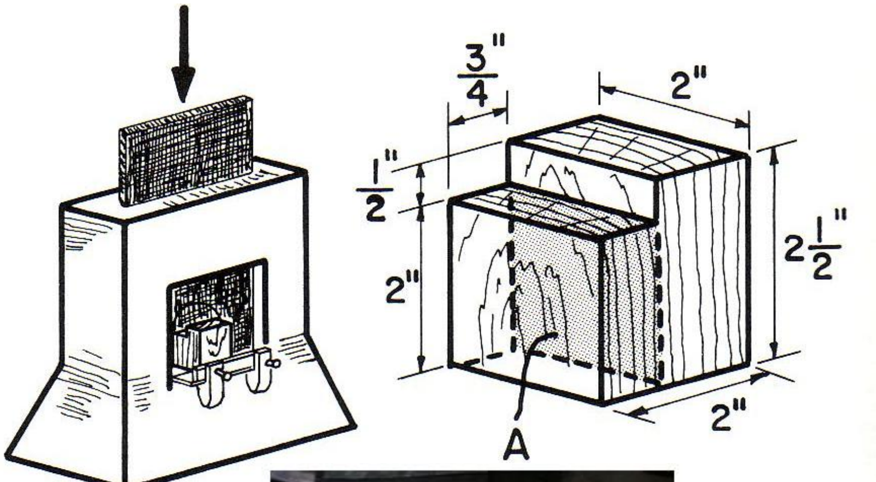
$$\sigma_{pl} = \frac{P_{pl}}{A}$$

Failure modes

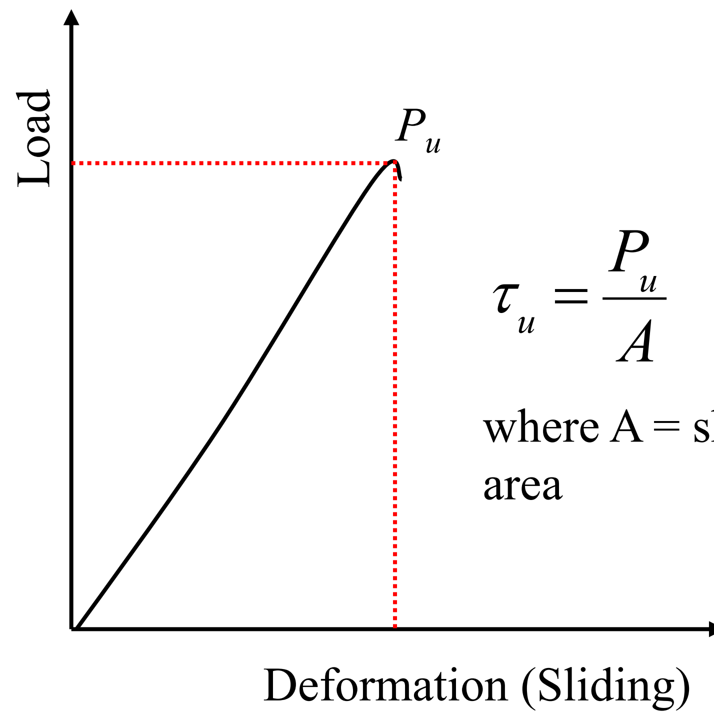


Failure types of clear wood in compression perpendicular to grain: (a) crushing of an earlywood zone, (b) shearing along a growth ring, (c) buckling of the growth rings.

Shear parallel to grain (Longitudinal shear)



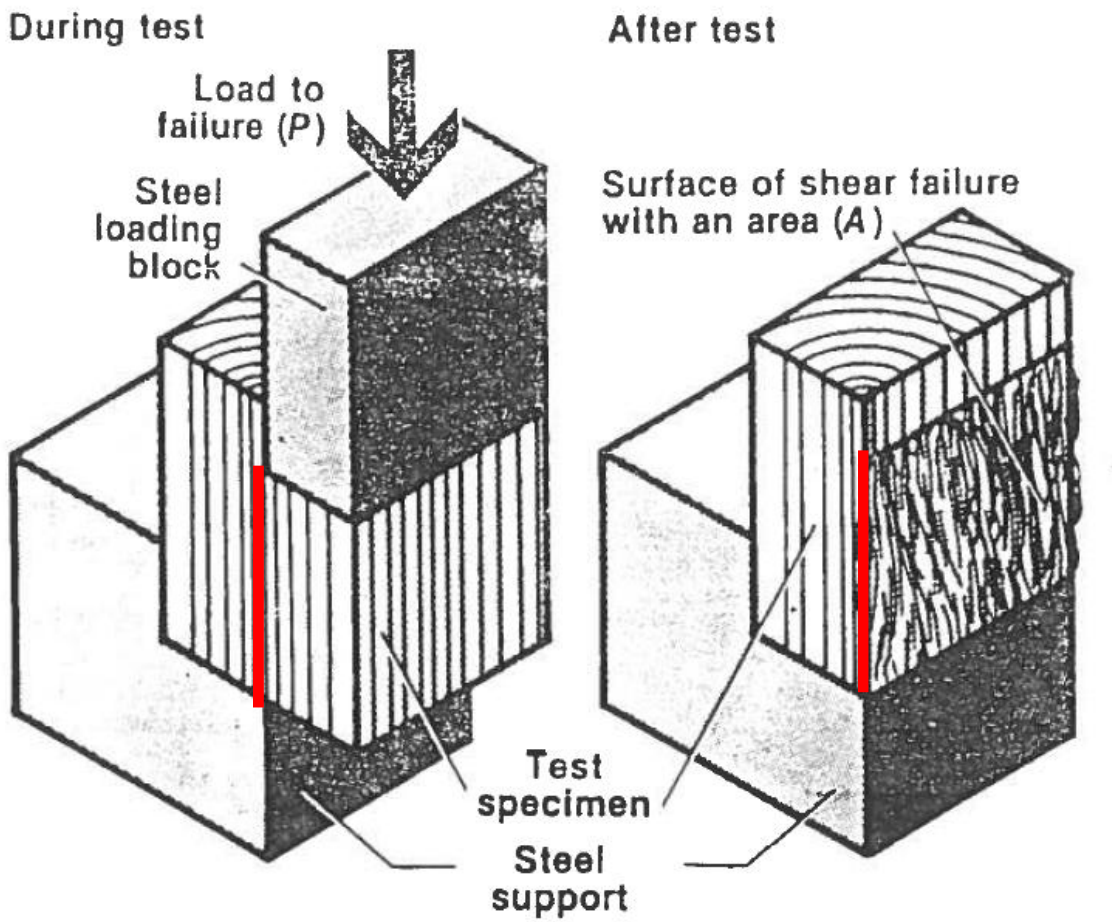
- Shear strength



$$\tau_u = \frac{P_u}{A}$$

where A = shear area

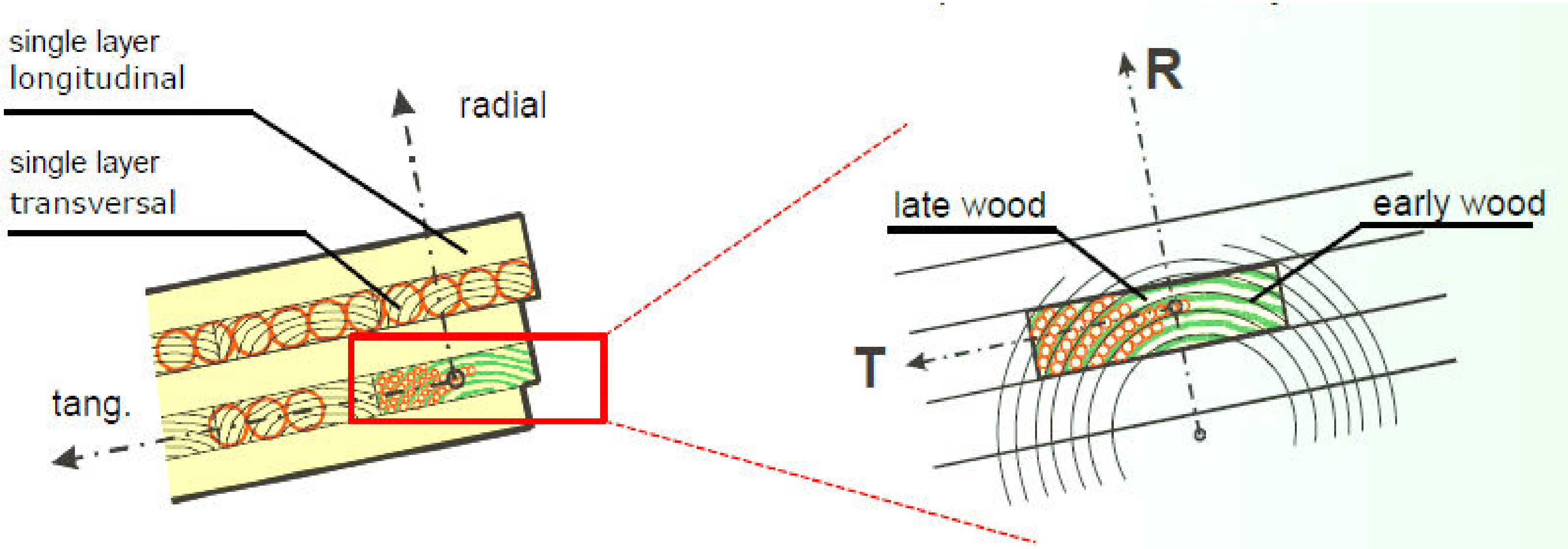
Failure mode



Shear through-thickness (rolling shear) properties of wood

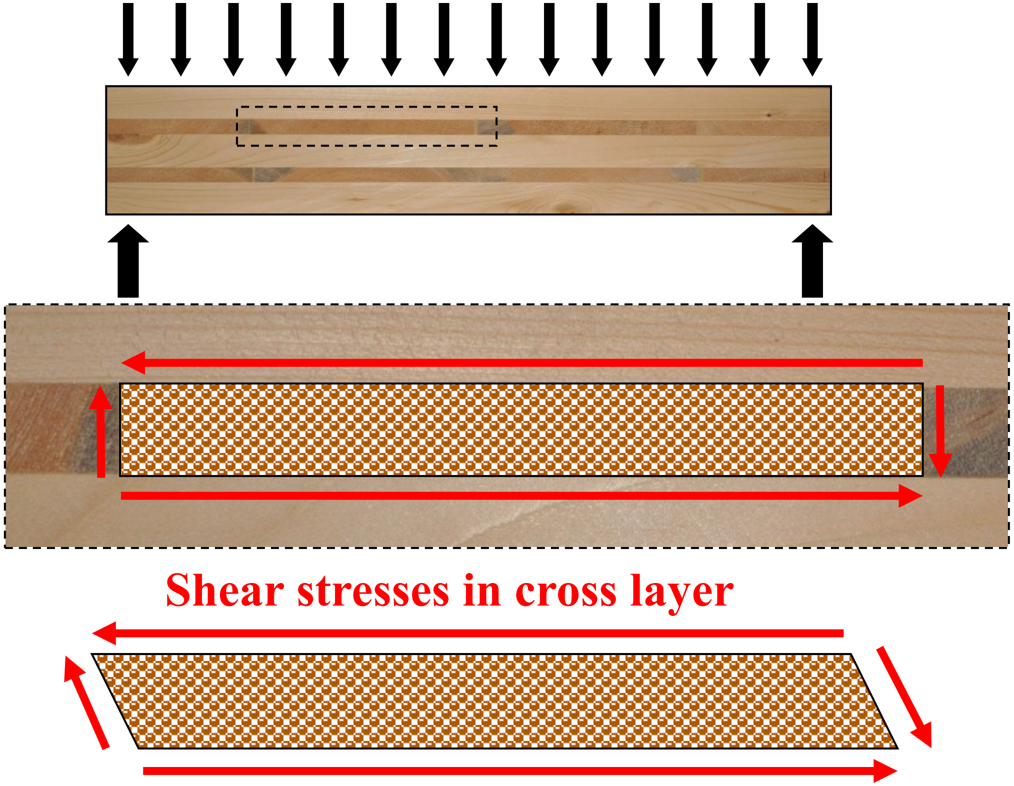
Weakness between growth rings leads to a tendency for wood layers to slide over one another under shear stress

Failure mode is for layers of wood to roll over – rolling shear

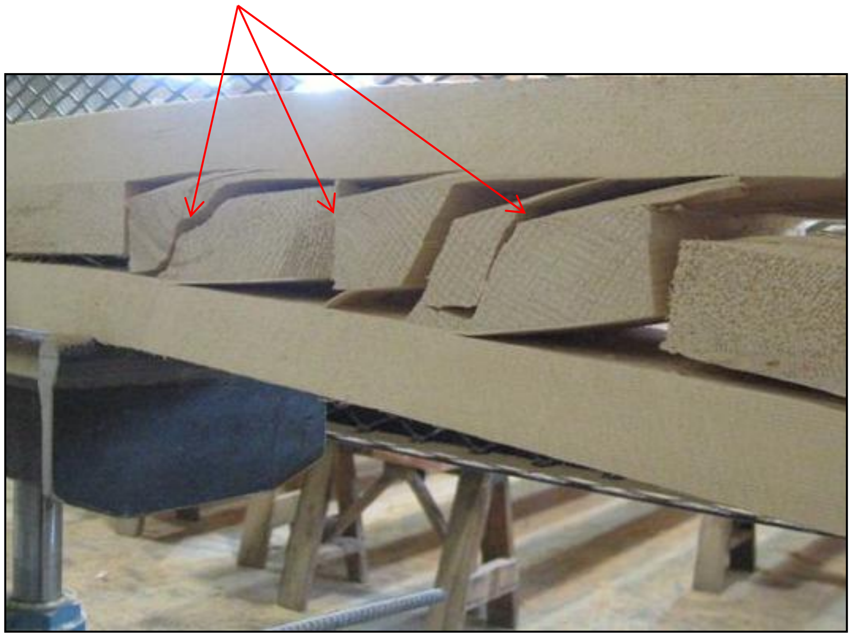


Rolling shear failure in CLT

- Under bending, failure could be rolling shear in a cross layer



Rolling Shear failure

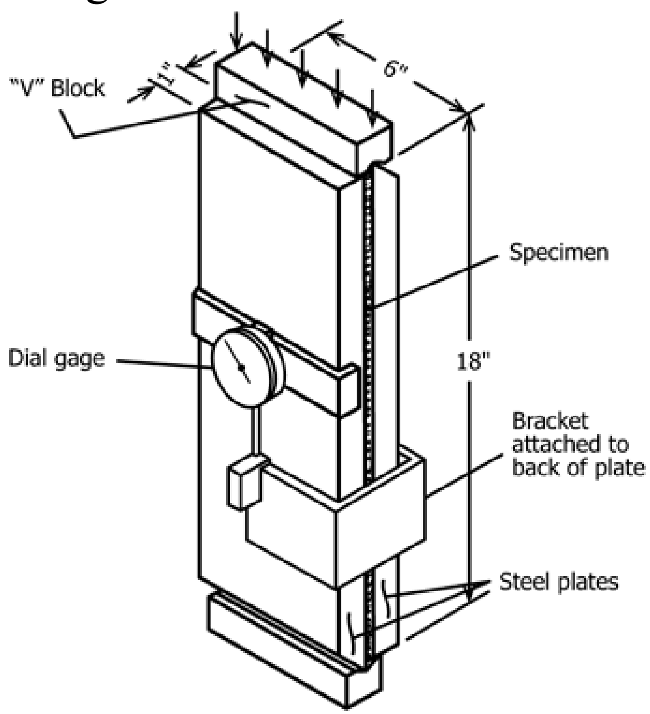


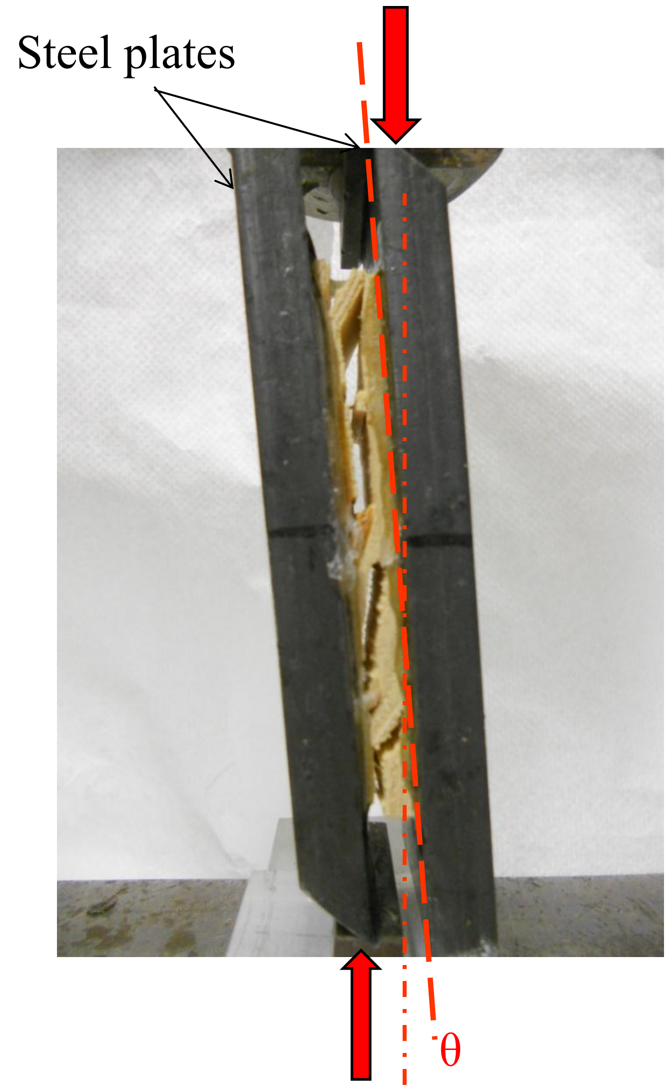
Measurement of rolling shear properties

Two-plate shear test method

Shear strength = $(P_{max} \cos \theta) / bL$

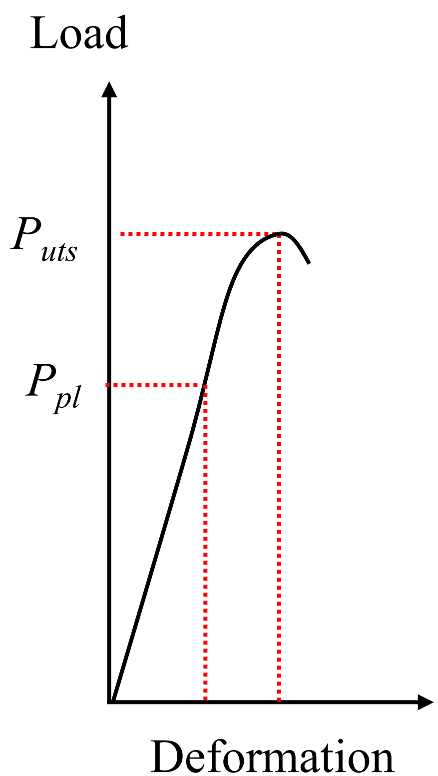
where b = width, L = length





Tension parallel to grain

- Tensile strength
- Modulus of elasticity
- Proportional limit

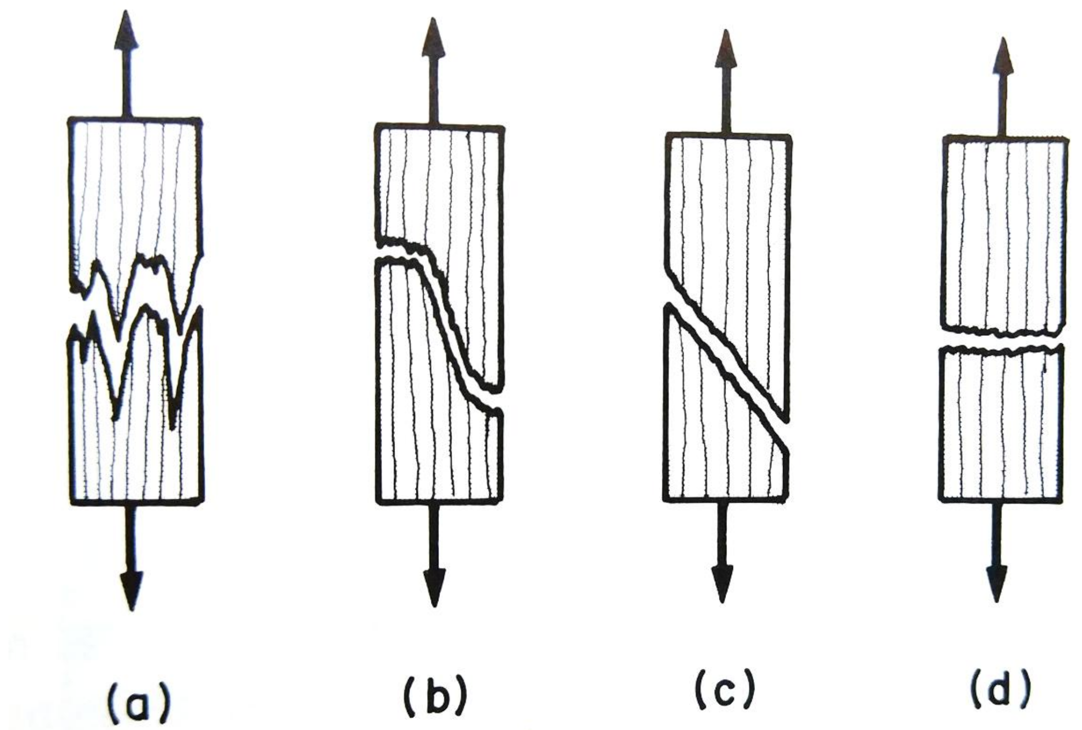


$$\sigma_{uts} = \frac{P_{uts}}{A}$$

$$\sigma_{pl} = \frac{P_{pl}}{A}$$

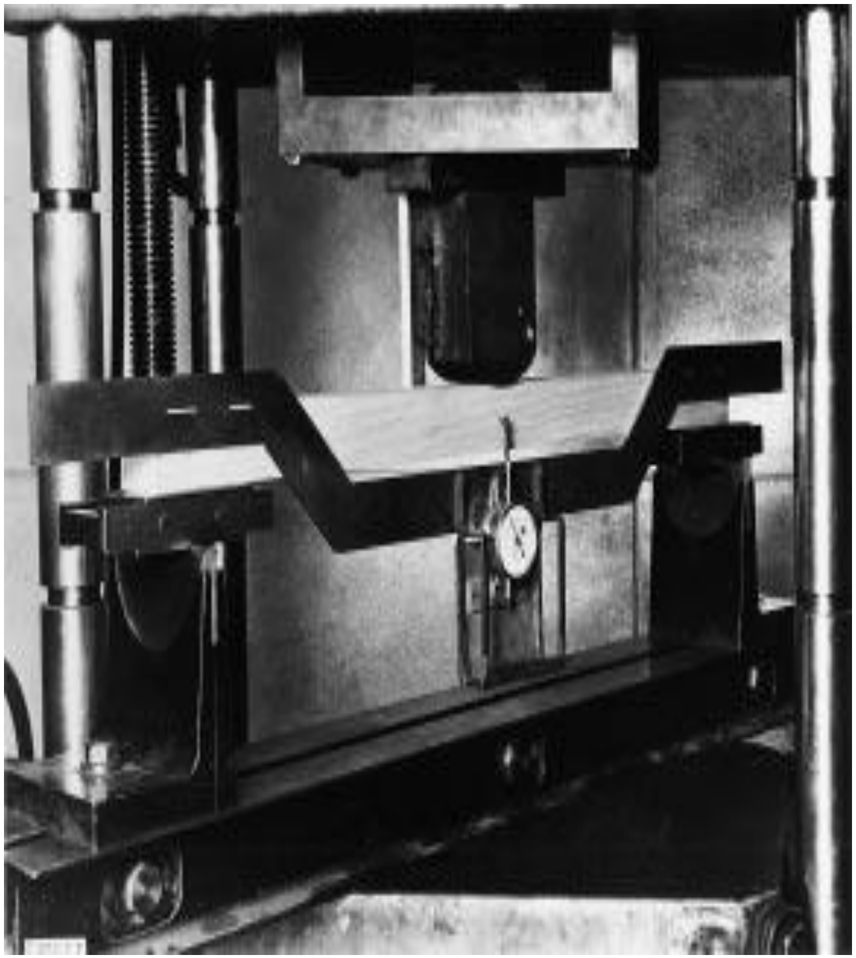
$$MOE = \frac{\sigma_{pl}}{\varepsilon_{pl}}$$

Failure mode



Failure types of clear wood in tension parallel to grain: (a) splintering tension, (b) combined tension and shear, (c) shear, (d) brittle tension.

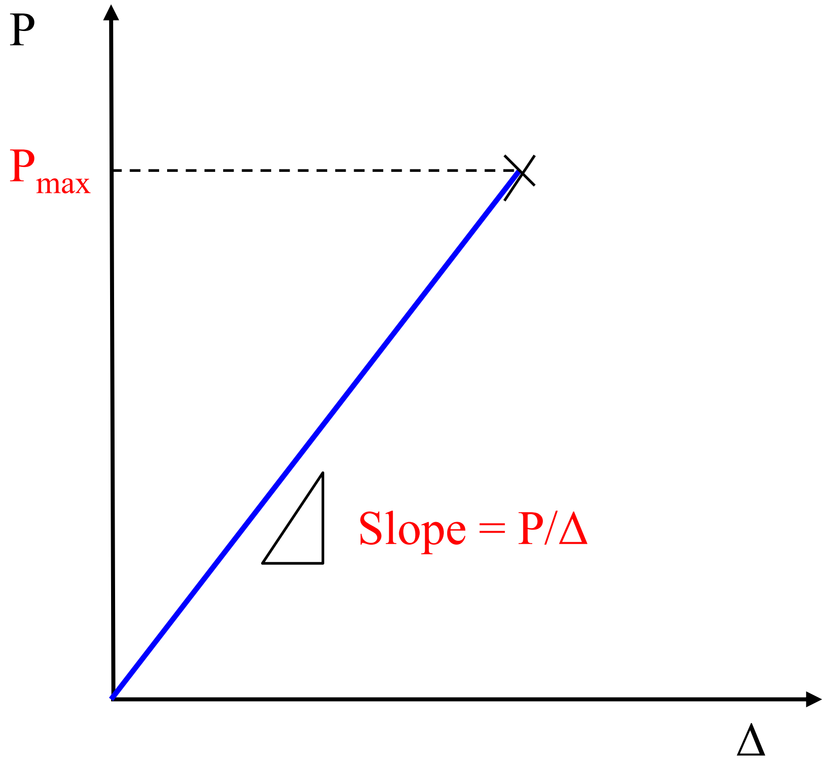
Bending

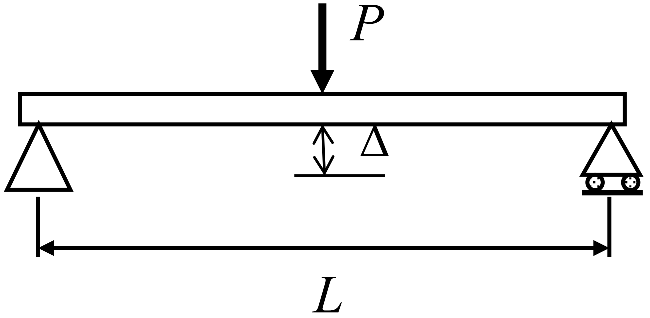


- Modulus of rupture (MOR)
- Modulus of elasticity (MOE)

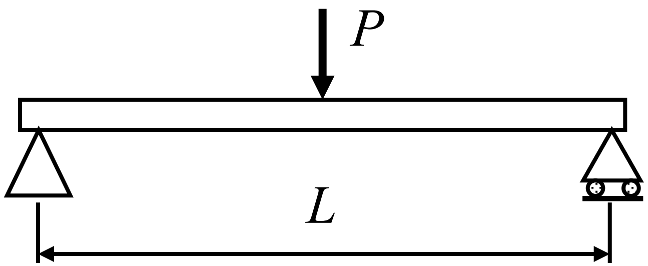
https://youtu.be/GnUpjeW8BT8

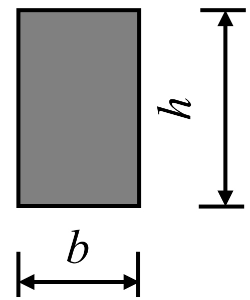
Load-deflection response in bending

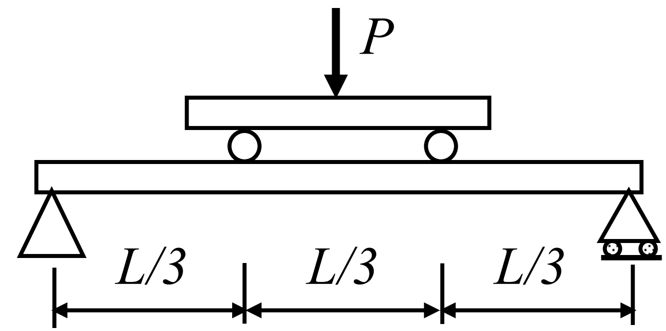




Bending properties of wood







Centre point loading

Third point loading

$$MOR = \frac{3P_{\max}L}{2bh^2}$$

$$MOR = \frac{P_{\max}L}{bh^2}$$

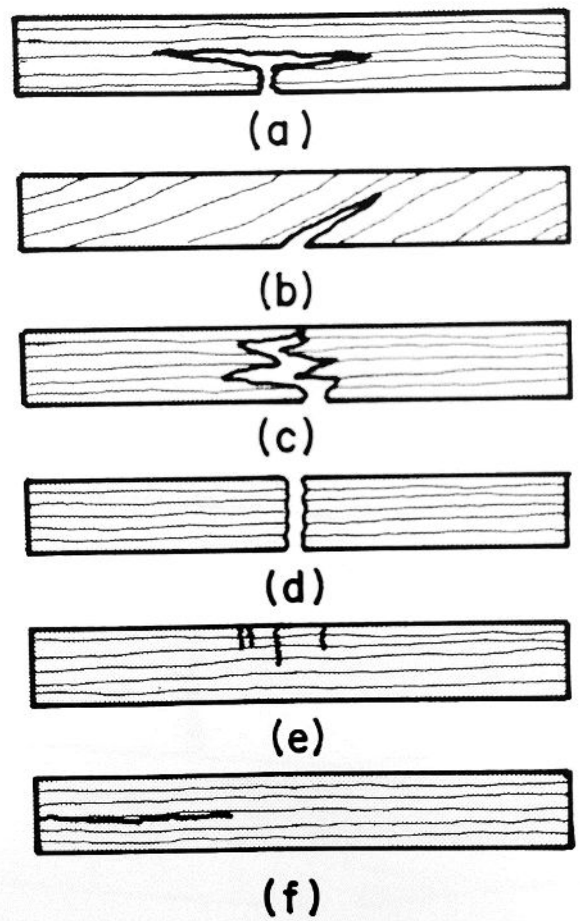
$$MOE = \left(\frac{P}{\Delta}\right)\frac{L^3}{4bh^3}$$

$$MOE = \left(\frac{P}{\Delta}\right)\frac{23L^3}{108bh^3}$$

P/Δ = slope of load-deformation line

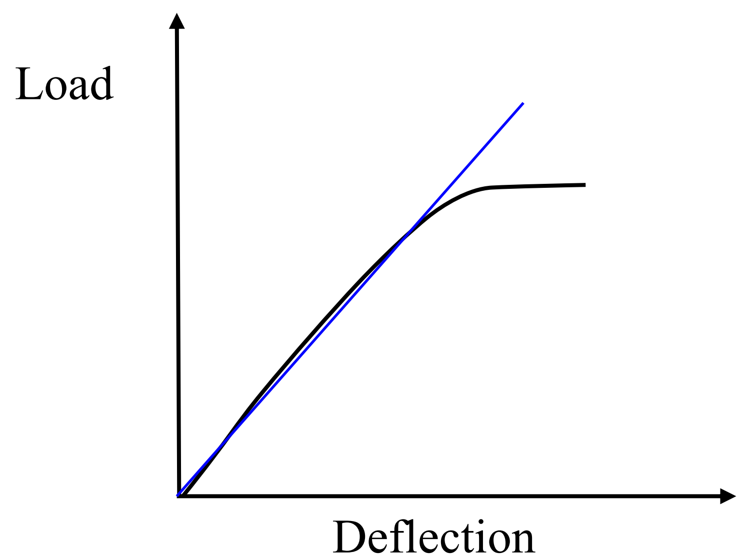
P_{max} = load at failure

Failure mode

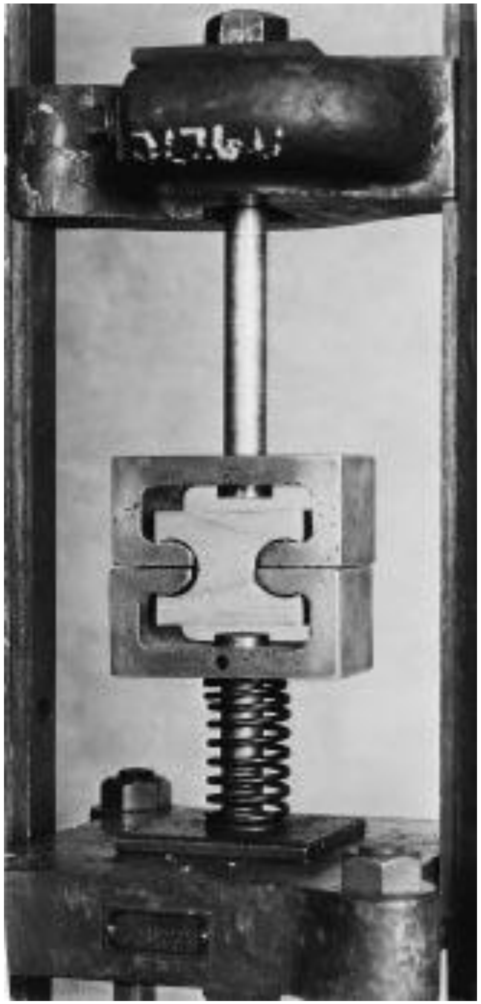


Failure types of clear wood in bending with span parallel to grain: (a) simple tension, (b) cross-grain tension, (c) splintering tension, (d) brash tension, (e) compression, (f) horizontal shear.

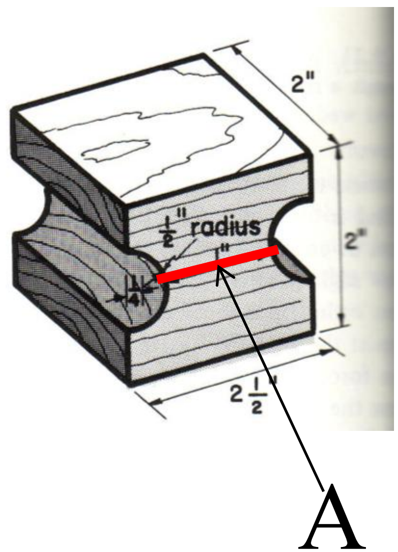
1. What causes each mode?
2. Which failure mode will provide the lowest failure load?
3. What failure mode may have non-linear response?



Tension perpendicular to grain



- Tensile strength
- Radial/tangential



$$\tau_u = \frac{P_u}{A}$$

Strength Requirements vs. Use

- Which strength property is important depends upon the end application.




Wood Handbook - Mechanical properties of wood

Table 5–4a. Mechanical properties of some commercially important woods grown in Canada and imported into the United States (metric)[a]

Common species names	Moisture content	Specific gravity	Static bending: Modulus of rupture (kPa)	Static bending: Modulus of elasticity (MPa)	Compression parallel to grain (kPa)	Compression perpendicular to grain (kPa)	Shear parallel to grain (kPa)
Hardwoods							
Aspen							
Quaking	Green	0.37	38,000	9,000	16,200	1,400	5,000
	12%		68,000	11,200	36,300	3,500	6,800
Big-toothed	Green	0.39	36,000	7,400	16,500	1,400	5,400
	12%		66,000	8,700	32,800	3,200	7,600
Cottonwood							
Balsam, poplar	Green	0.37	34,000	7,900	14,600	1,200	4,600
	12%		70,000	11,500	34,600	2,900	6,100
Black	Green	0.30	28,000	6,700	12,800	700	3,900
	12%		49,000	8,800	27,700	1,800	5,900
Eastern	Green	0.35	32,000	6,000	13,600	1,400	5,300
	12%		52,000	7,800	26,500	3,200	8,000

Small clear wood with straight grain

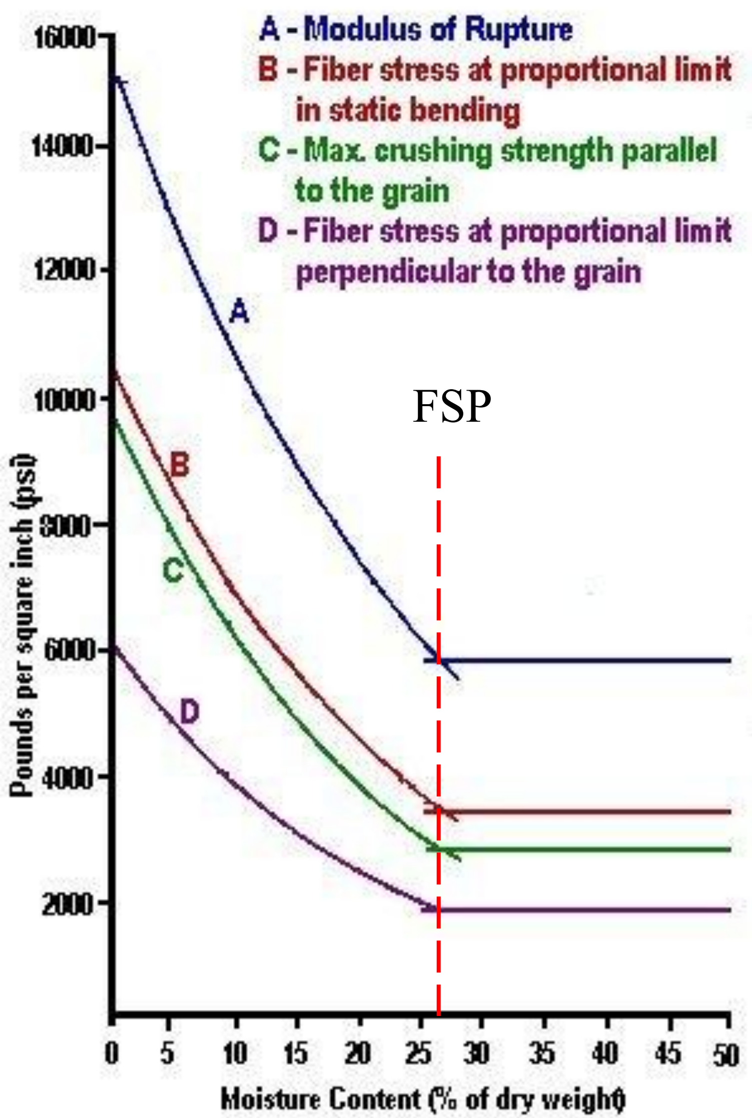
Notes on interpreting test data from literature

- Small clear wood vs structural size lumber
- Moisture condition of wood
- Origin and growth conditions of trees – geographical locations
- Graded vs ungraded wood

Mechanical properties

- Behaviour of wood under different types of stress
- Methods of measurement for small clear wood
- Factors influencing mechanical properties of wood
  - Environmental (e.g. moisture, temperature)
  - Growth features (e.g. grain, knots, density)
  - Long-term loading

Moisture content



- Fibre saturation point (FSP) is MC above which mechanical properties are not affected by MC
- Slope - Similar among strength properties

Adjustment factor (α) of moisture content for clear wood

Table 2.4 Average changes in mechanical properties of clear wood due to one percent change in moisture content (based on data of Bodig and Jayne (1982) and Hoffmeyer (1995))

Property	Change (%)
Compressive strength parallel	5
Compressive strength perpendicular	5.5
Shear strength parallel	3
Modulus of rupture parallel	4
Modulus of elasticity parallel	2
Tension strength parallel	2.5
Tension strength perpendicular	1.5

$$\alpha = \frac{(P_1 - P_2)/P_1}{\Delta M}$$

- P_2: property at M_2 moisture content
- P_1: property at M_1 moisture content
- ΔM: change in moisture content, $= (M_2 - M_1) > 0$
- Assuming linear between 0 and 25% moisture content.

Estimation of 'strength' of clear wood at a given moisture content

$$P_M = P_{12}\left(\frac{P_{12}}{P_g}\right)^{\left(\frac{12-M}{M_p-12}\right)}$$

P_M : *property at* $M\%$ *moisture content*

P_{12} : *property at* 12% *moisture content*

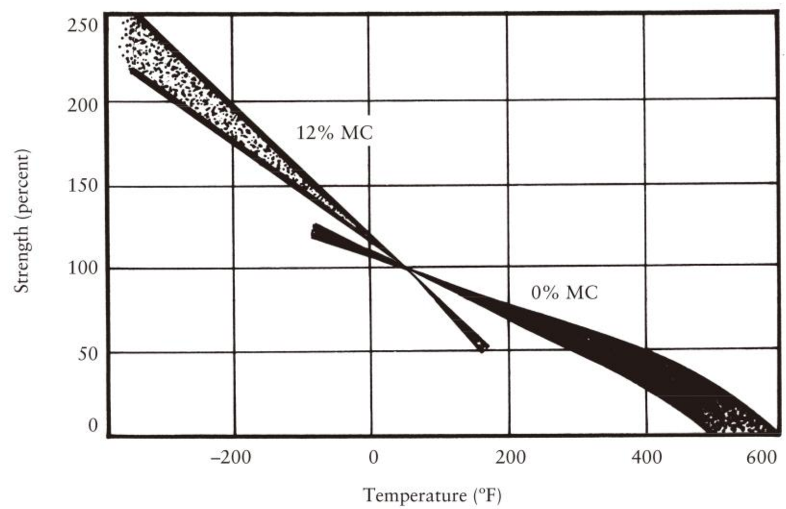
P_g : *property at green condition*

M : *moisture content in percentage*

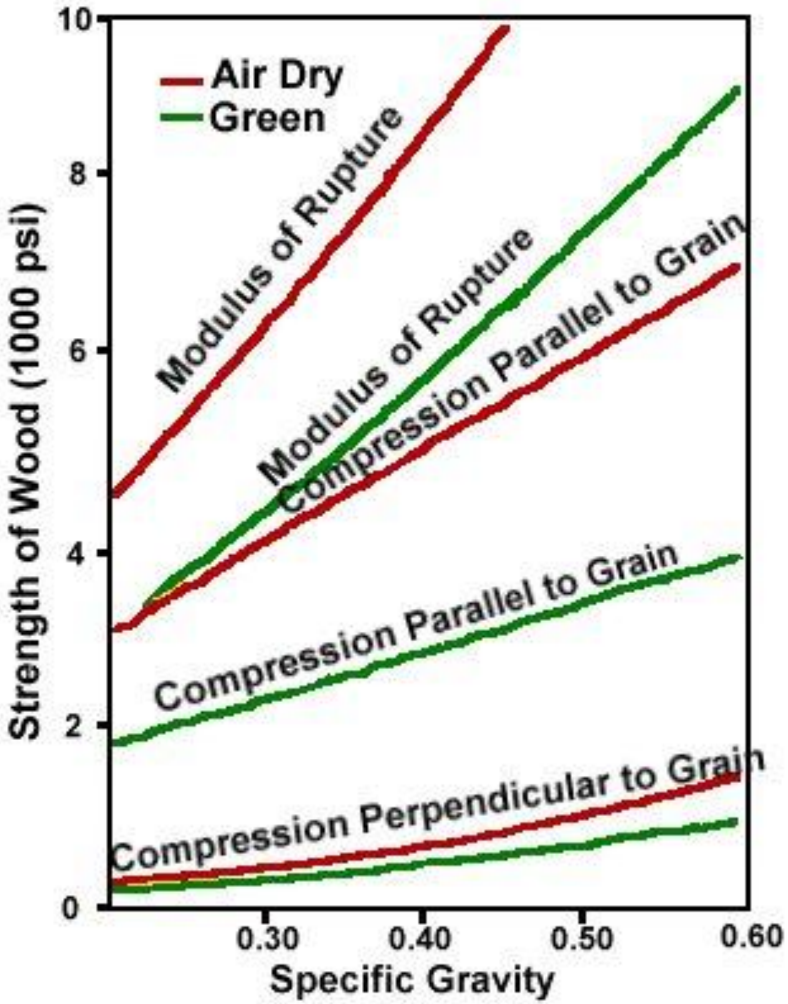
M_p : *a species – dependent value which is slightly lower than FSP and is usually taken at* 25%

Effect of temperature

- Mechanical properties decrease when heated and increase when cooled
- Change is dependent on MC of wood (note: when wood is heated MC changes also)
- Below 100° C the change is reversible ie no permanent change in characteristics



Specific gravity and mechanical properties



- Mechanical properties and Specific Gravity are correlated

$$P = a\,G^{b}$$

where

P: predicted property

G : specific gravity

a and *b*: constants dependent upon property

- Relationship depends on MC, properties and SW/HW

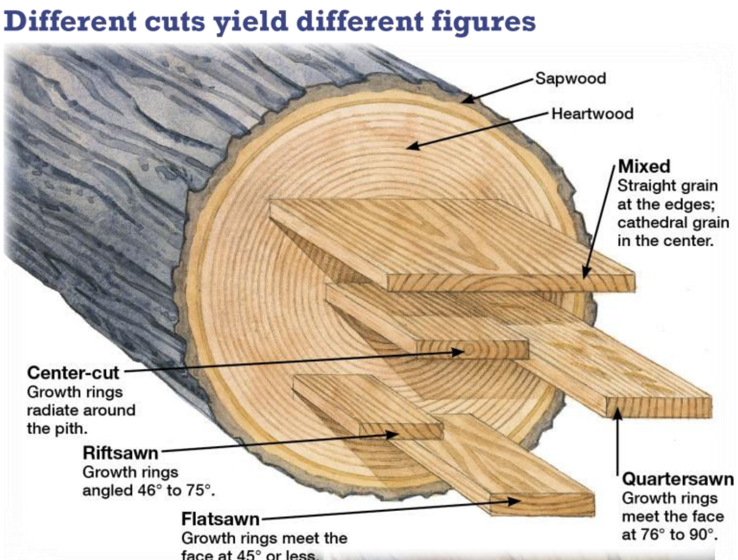
Wood Handbook

Table 5–11a. Functions relating mechanical properties to specific gravity of clear, straight-grained wood (metric)

	Specific gravity–strength relationship			
	Green wood		Wood at 12% moisture content	
Property[a]	Softwoods	Hardwoods	Softwoods	Hardwoods
Static bending				
MOR (kPa)	$109,600\ G^{1.01}$	$118,700\ G^{1.16}$	$170,700\ G^{1.01}$	$171,300\ G^{1.13}$
MOE (MPa)	$16,100\ G^{0.76}$	$13,900\ G^{0.72}$	$20,500\ G^{0.84}$	$16,500\ G^{0.7}$
WML (kJ m^{-3})	$147\ G^{1.21}$	$229\ G^{1.51}$	$179\ G^{1.34}$	$219\ G^{1.54}$
Impact bending (N)	$353\ G^{1.35}$	$422\ G^{1.39}$	$346\ G^{1.39}$	$423\ G^{1.65}$
Compression parallel (kPa)	$49,700\ G^{0.94}$	$49,000\ G^{1.11}$	$93,700\ G^{0.97}$	$76,000\ G^{0.89}$
Compression perpendicular (kPa)	$8,800\ G^{1.53}$	$18,500\ G^{2.48}$	$16,500\ G^{1.57}$	$21,600\ G^{2.09}$
Shear parallel (kPa)	$11,000\ G^{0.73}$	$17,800\ G^{1.24}$	$16,600\ G^{0.85}$	$21,900\ G^{1.13}$
Tension perpendicular (kPa)	$3,800\ G^{0.78}$	$10,500\ G^{1.37}$	$6,000\ G^{1.11}$	$10,100\ G^{1.3}$
Side hardness (N)	$6,230\ G^{1.41}$	$16,550\ G^{2.31}$	$8,590\ G^{1.49}$	$15,300\ G^{2.09}$

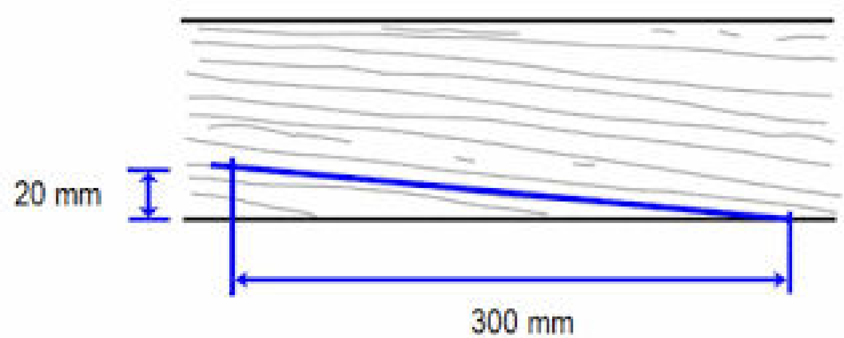
[a]Compression parallel to grain is maximum crushing strength; compression perpendicular to grain is fiber stress at proportional limit. MOR is modulus of rupture; MOE, modulus of elasticity; and WML, work to maximum load. For green wood, use specific gravity based on ovendry weight and green volume; for dry wood, use specific gravity based on ovendry weight and volume at 12% moisture content. Calculated using all data from Table 5–3.

Slope of grain (SoG) in lumber product



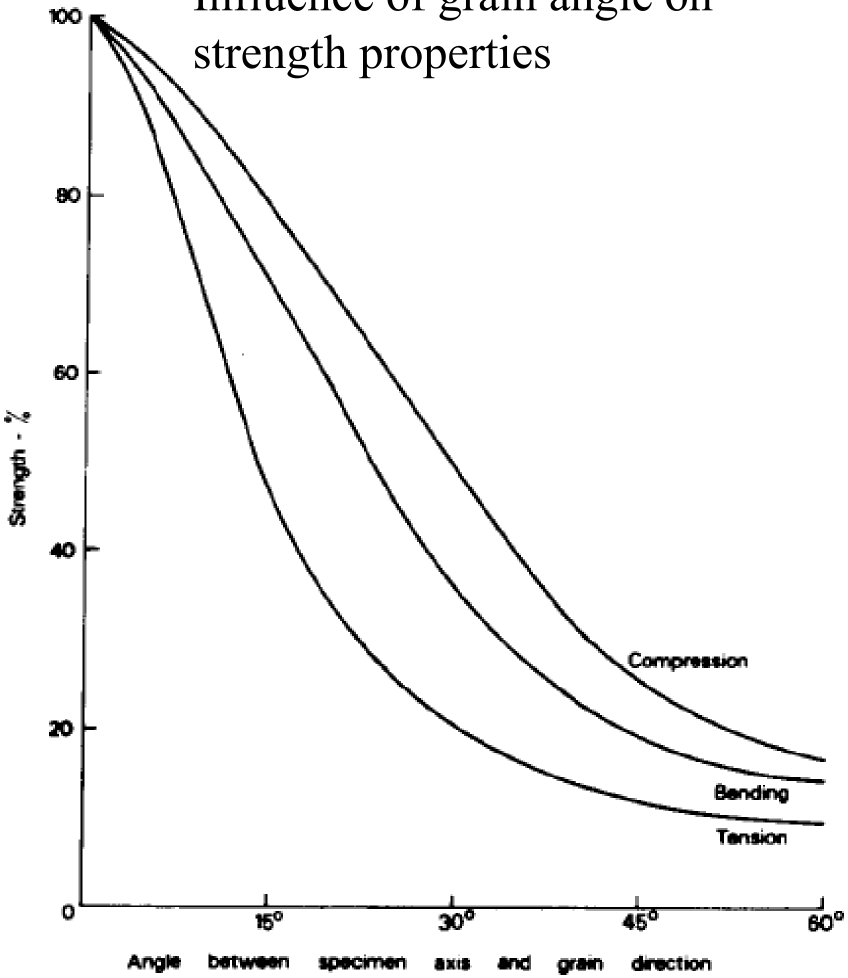
(Source: woodmagazine.com)

- SoG is one of the most important growth features affecting mechanical properties of lumber.
- Arise from:
  - When lumber is sawn not following the direction of the cell in the log
  - Cells are aligned in living tree in a spiral grain pattern

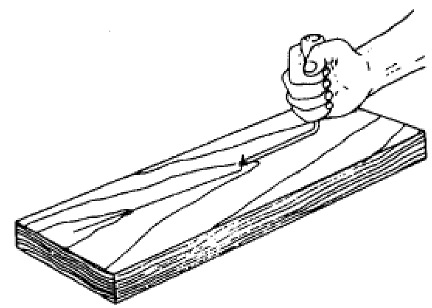


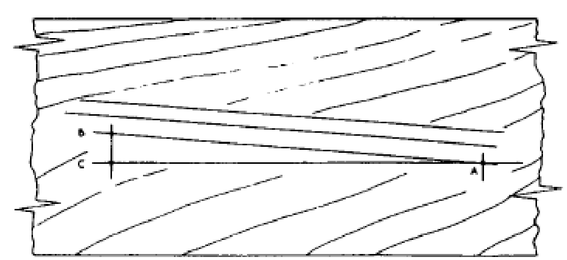
Measurement of grain angle

Influence of grain angle on strength properties



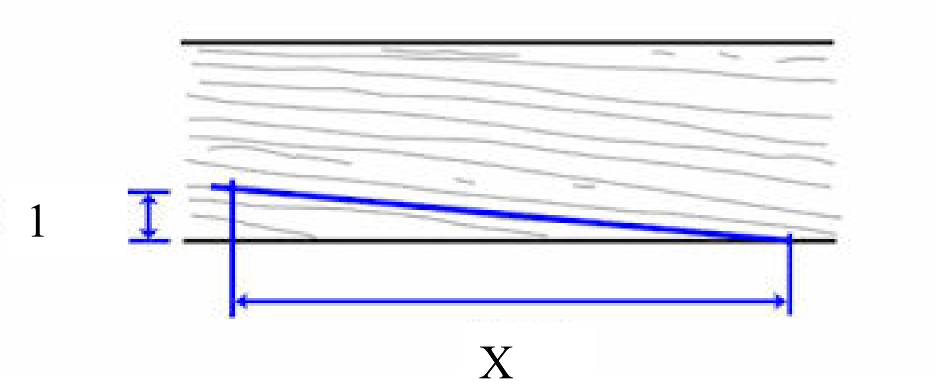
- Measurement of grain angle in lumber
  - Grain scribe
  - Scanning machine





Slope of Grain

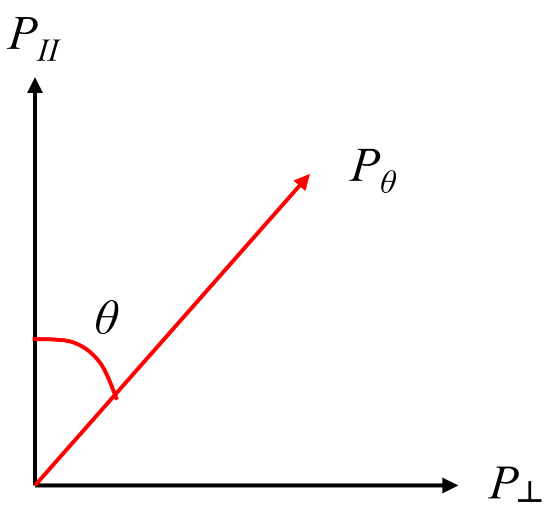
Slope of Grain (1 in X)	Strength/ Straight grain strength
0	100%
1 in 20	93%
1 in 10	81%
1 in 5	55%



Hankinson's formula

$$P_\theta = \frac{P_{//} P_\perp}{P_{//} \sin^n \theta + P_\perp \cos^n \theta}$$

- P_θ: property when wood is loaded at an angle θ to the grain direction
- P_{II}: property when wood is loaded parallel to grain (θ=0°)
- $P_\perp$: property when wood is loaded perpendicular to grain (θ=90°)
- n: property-dependent constant

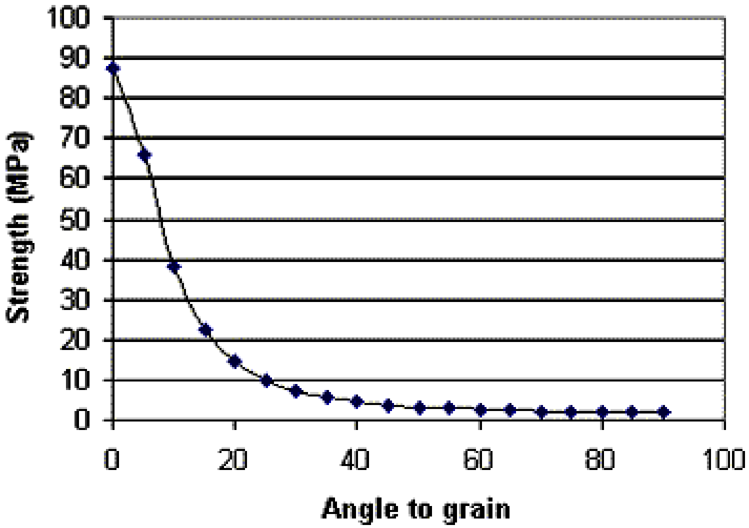


Property	*n*	$P_\perp / P_{//}$
Tension	1.5-2.0	0.04-0.07
Compression	2.0-2.5	0.03-0.40
MOR	1.5-2.0	0.04-0.10
MOE	2.0	0.04-0.12

Example

Known:
Species: Douglas Fir
Parallel tensile strength = 87.6 MPa
Perpendicular tensile strength = 2 MPa

Question: Tensile strength at 45 degrees?



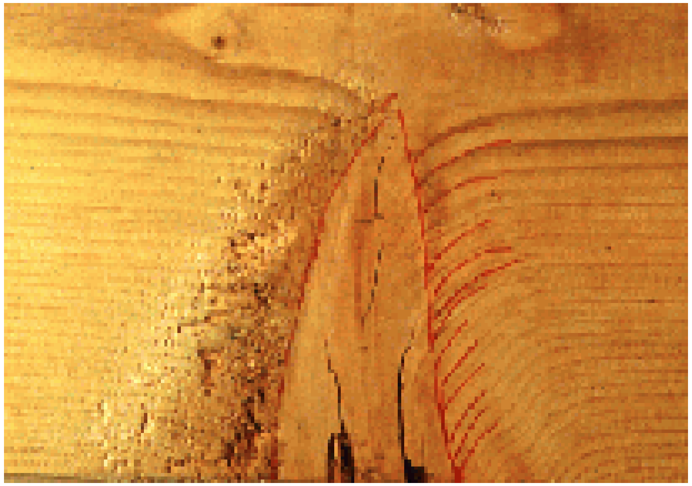
Solution: Hankinson's formula,
let n=2

$$\sigma_{45} = \frac{\sigma_{\parallel}\sigma_{\perp}}{\sigma_{\parallel}\sin^2 45 + \sigma_{\perp}\cos^2 45}$$

$$= \frac{87.6 \times 2}{87.6 \times (\frac{\sqrt{2}}{2})^2 + 2 \times (\frac{\sqrt{2}}{2})^2} = 3.91\,MPa$$

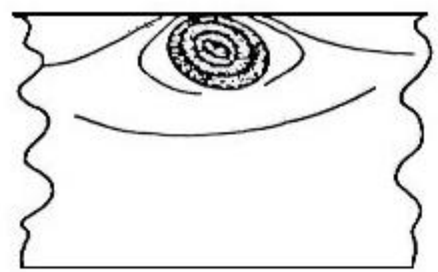
Knots

- **Nature**
  - intergrown or tight
  - encased or loose
- **Shape**
  - Round
  - Spike
  - Oval



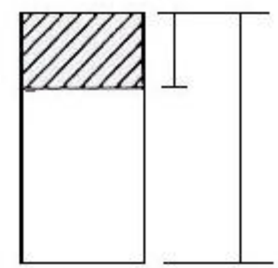
Knots

Effect of Knots or Holes on Strength of Bending Members (Bending Strength as a Percent of Knot-Free Wood)				
Knot Diameter (inches)	Knot or Hole on Tension Edge or Wide Face		Knot or Hole on Neutral Axis	
	Nominal 2 × 10	Nominal 2 × 6	Nominal 2 × 10	Nominal 2 × 6
0	100	100	100	100
1/2	91	85	95	92
1	82	70	90	84
2	65	45	80	67
3	50	21	70	50



A

An edge knot in lumber



B

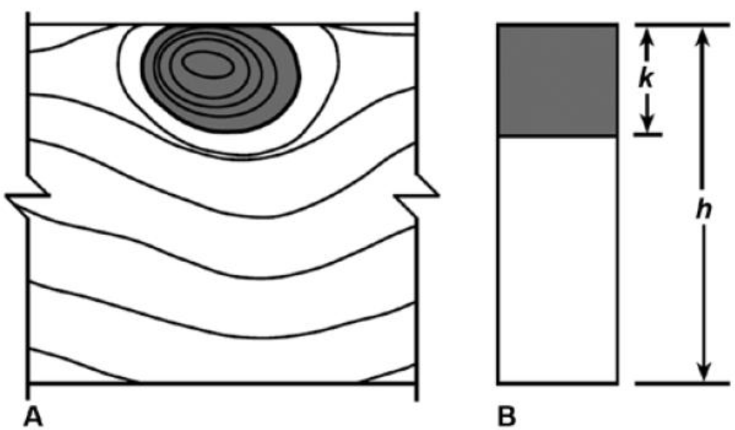
the assumed loss of cross section

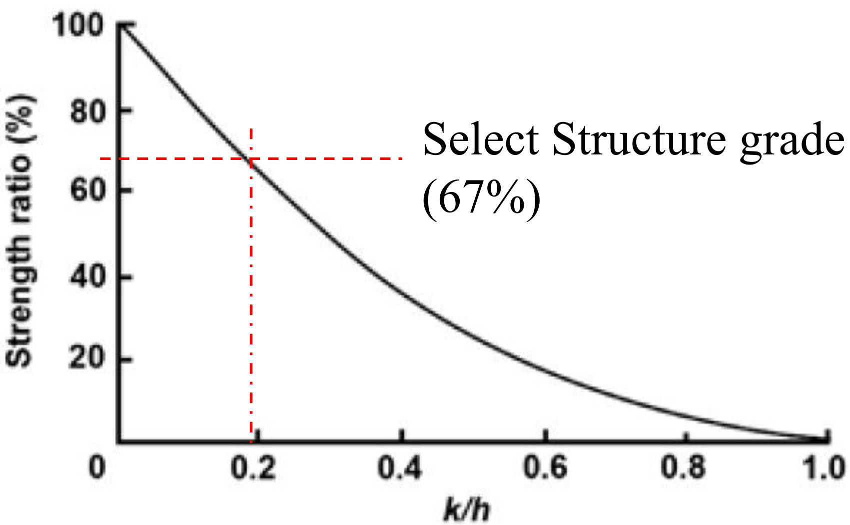
Source : Wood Handbook

Strength ratio concept

$$\text{Strength ratio} = \frac{\text{Strength of lumber with defects}}{\text{Strength of clear wood with straight grain}}$$

- Form the basis of visual grading rules





Influence of edge knot on strength ratio

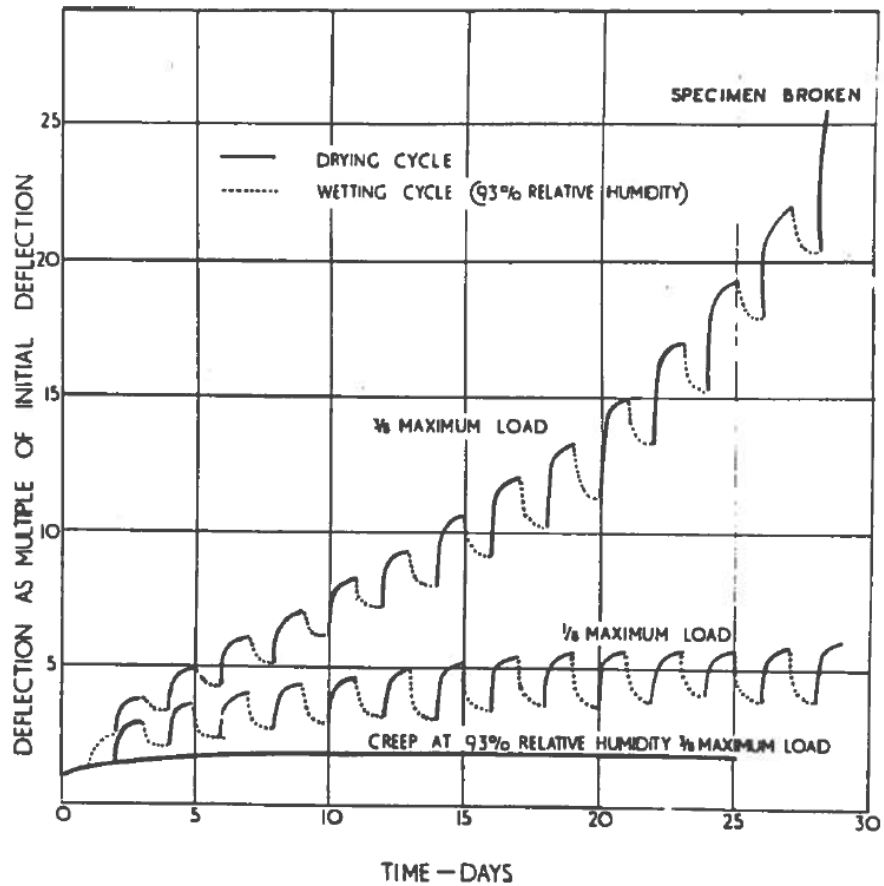
Source : Wood Handbook

Creep and creep rupture of wood

- Wood continues to deform under the action of a constant sustained load – Creep
- Wood subjected to a sustained load may fail over time – Creep rupture (also known as duration of load)

Creep phenomenon

- Creep is increased by fluctuating moisture condition
- Long-term deflection needs to be accounted for under dead load in design



Effect of relative humidity cycling on the creep behavior of beech in bending at two load levels.

Creep rupture

- Wood can carry a higher load for a short period of time than it can carry for a long period of time
- Basis of duration of load effect factor in design standards is the Madison curve developed at US Forest Products Lab

Madison curve

Applied load ratio

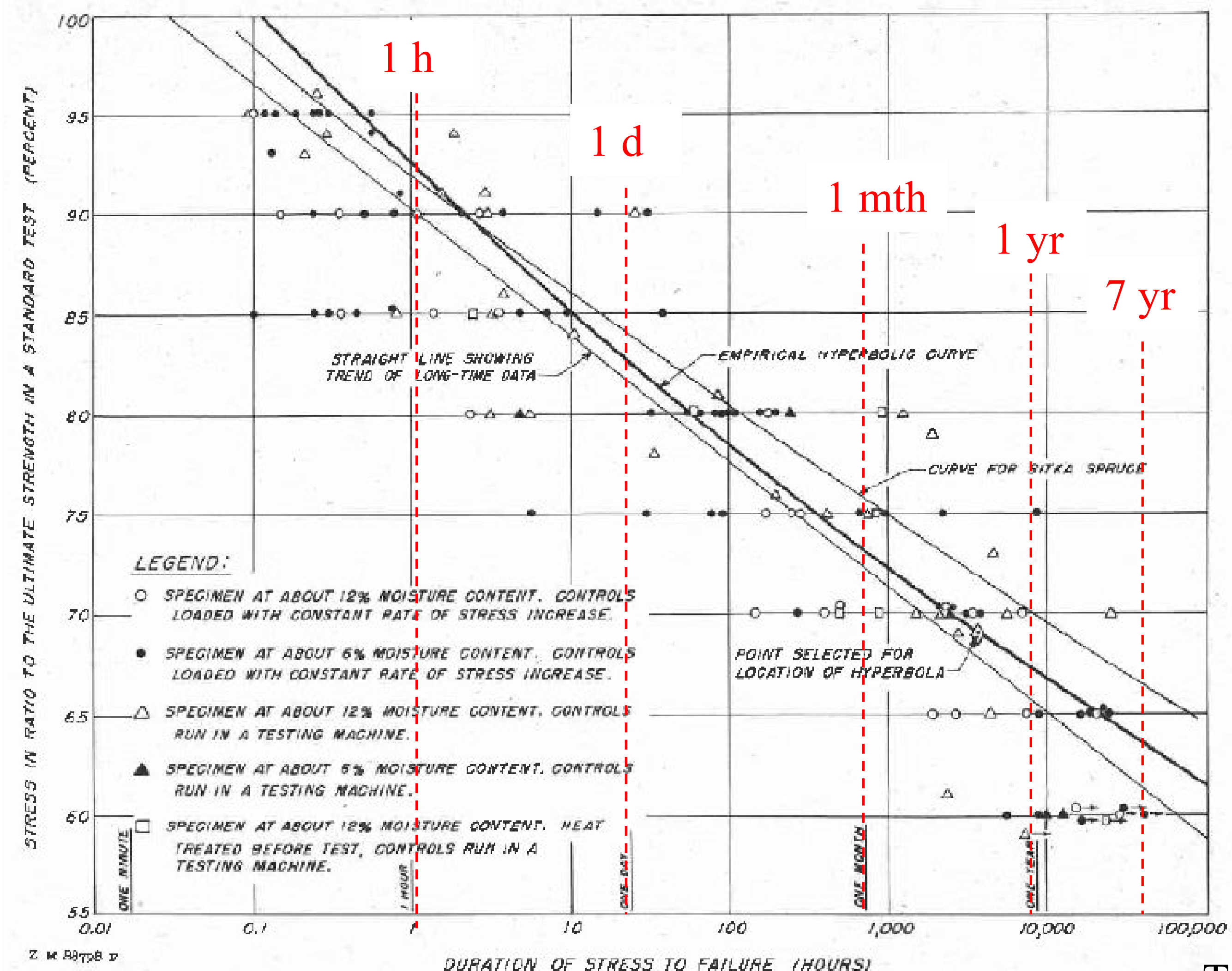


Figure 1.--Relation of duration of constant stress to level of stress in long-time loading of Douglas-fir bending specimens.

Time (h)

- Groups of lumber tested at different load levels relative to short-term strength
- Load held constant until failure
- Longest duration is 7 years

Typical load duration factors for design

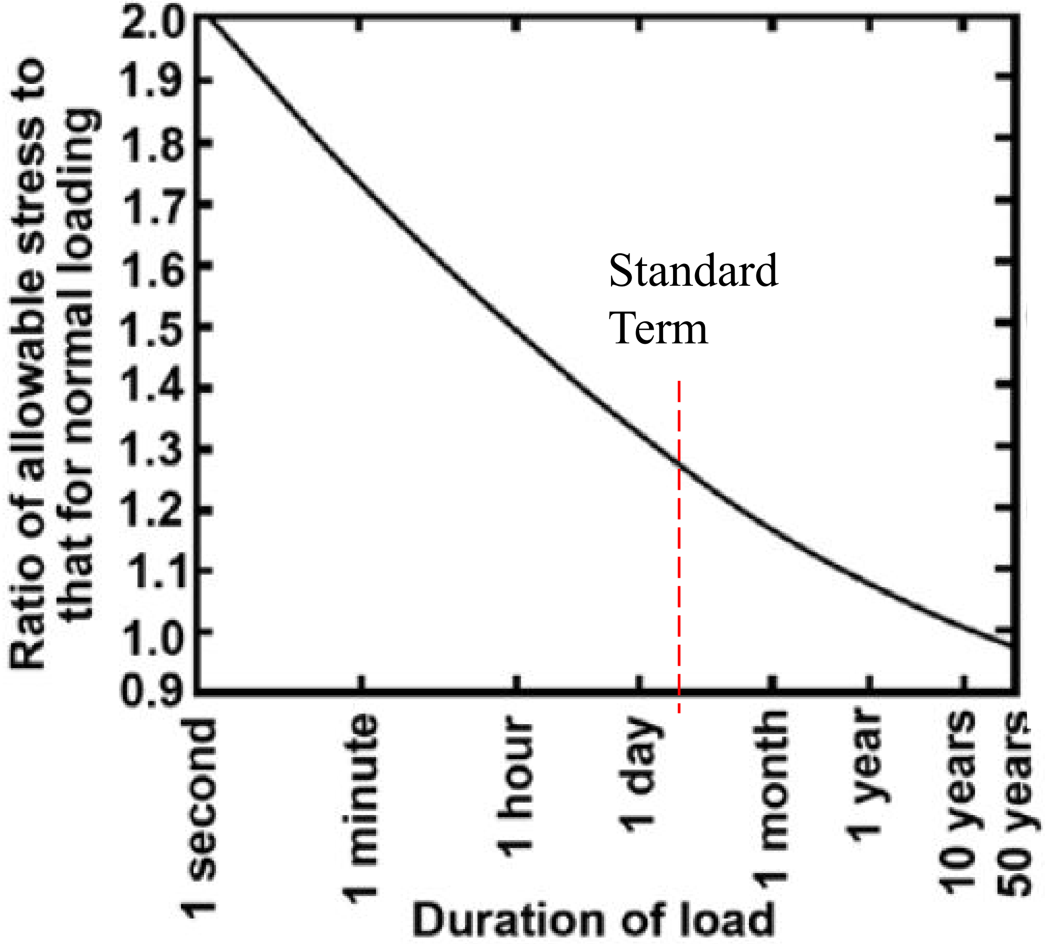


Figure 7–9. Relation of strength to duration of load.

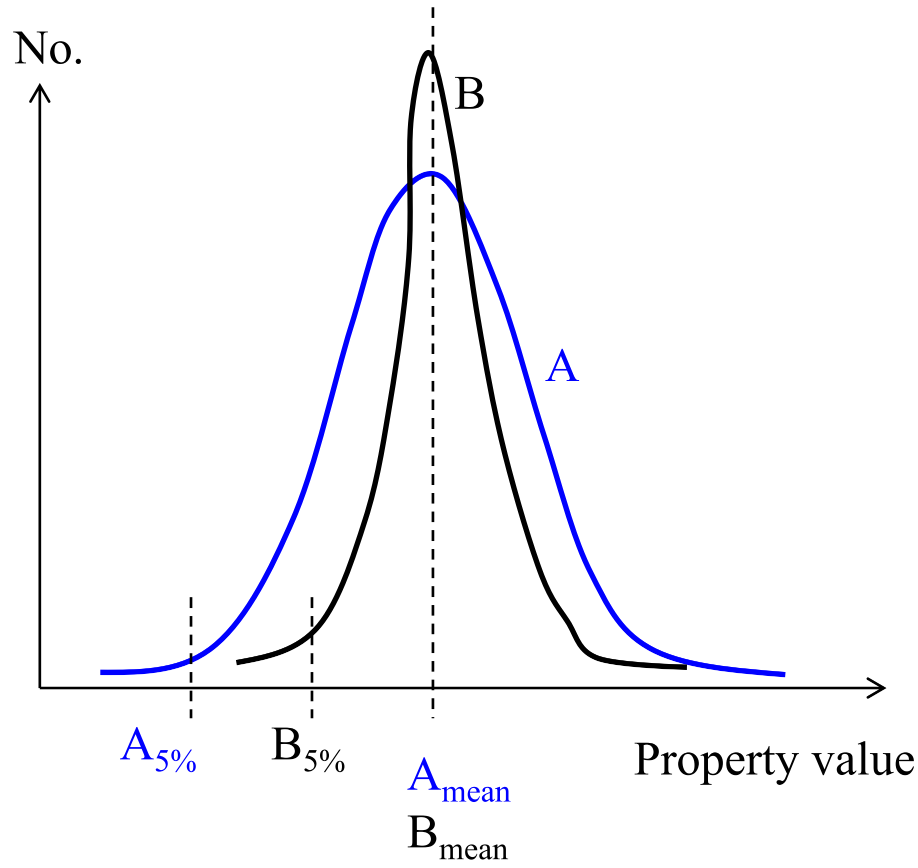
K_D :
Short-term 1.15
Standard term 1.0
Long term 0.65

Derivation of design properties

- Design properties are derived based on either
  - Testing (lumber, SCL, panels); or
  - Calculation (wood I-joist, glulam, CLT)
- Strength design properties (e.g. MOR) are based on 5th percentile with a 75 percent confidence
- Stiffness design properties (e.g. MOE) are based on mean value

Derivation of design properties

- Mechanical properties of wood products have high variability – relatively large sample size is required (>30)
- A statistical distribution is fitted to test data (e.g. Weibull, Normal)
- Product B will have a higher strength design property than Product A (a lower variability but similar mean value)



Derivation of design properties

- For Limit States Design code,

Specified strength = $A_{5\%}$ x K_S x K_{DoL} x K_{MC}

where

K_S = other factors that may be necessary

K_{DoL} = Duration of load factor to convert from short-term test (5 min) to standard term (roughly 7 days)

K_{MC} = Moisture adjustment factor to convert to 15% MC level (dry condition definition), if test condition differs

Design modification factors for timber structures

Modification factors for structural timber design

- Duration of load factor K_D
  - To reflect the effect of load duration on strength
- Service condition factor K_S
  - To take into account the effect of MC on strength
- Size factor K_Z
  - To take into account the effect of size on strength
- System factor K_H
  - To take into account the system behaviour
- Treatment factor K_T
  - To account for the influence of preservative and fire-retardant treatment

Duration of Load Effect

- Wood can carry a higher load for a short period of time than it can carry for a long period of time
- Basis of duration of load effect factor in design standards is the Madison curve developed at US Forest Products Lab

Modification Factors

Load duration factor, K_D

- Short term (< 7 days), K_D=1.15 (wind, earthquake, formwork, impact)
- Standard term, K_D=1.0 (snow, live, dead load in combination with snow or live)
- Long term, K_D= 0.65 (dead load, earth pressure, retaining wall)

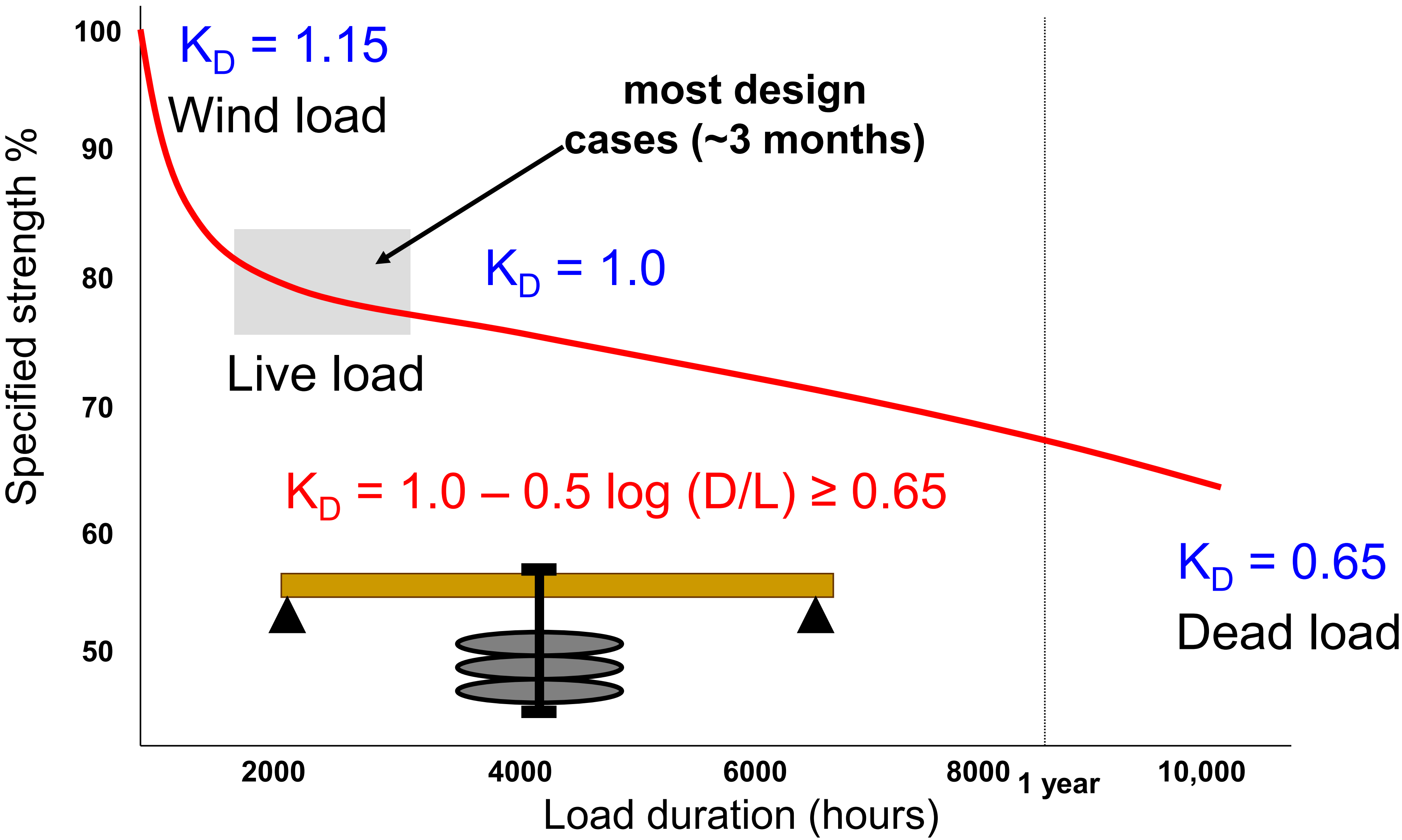
Duration of load factor, K_D

Table 4.3.2.2
Load duration factor, K_D

Load duration	K_D	Explanatory notes
Short term	1.15	Short-term loading means the condition of loading where the duration of the specified loads is not expected to last more than 7 days continuously or cumulatively throughout the life of the structure. Examples include wind loads, earthquake loads, falsework, and formwork, as well as impact loads.
Standard term	1.00	Standard term means the condition of loading where the duration of specified loads exceeds that of short-term loading, but is less than long-term loading. Examples include snow loads, live loads due to occupancy, wheel loads on bridges, and dead loads in combination with all of the above.
Long term	0.65	Long-term duration means the condition of loading under which a member is subjected to more or less continuous specified load. Examples include dead loads or dead loads plus live loads of such character that they are imposed on the member for as long a period of time as the dead loads themselves. Such loads include those usually occurring in tanks or bins containing fluids or granular material, loads on retaining walls subjected to lateral pressure such as earth, and floor loads where the specified load can be expected to be continuously applied, such as those in buildings for storage of bulk materials. Loads due to fixed machinery should be considered to be long term.

Note: *Load duration requires professional judgment by the designer. Explanatory notes in this Table provide guidance to designers about the types of loads and load combinations for which each modification factor should be applied.*

Duration of load effect, K_D



Modification Factors - K_D

4.3.2.3 Long-term load factor

For standard-term loads where the specified long-term load, P_L, is greater than the specified standard-term load, P_S, the long-term load factor may be used or the factor may be calculated as follows:

$$K_D = 1.0 - 0.50\log(P_L/P_S) \geq 0.65$$

where

P_L = specified long-term load

P_S = specified standard-term load based on S and L loads acting alone or in combination

= S, L, $S + 0.5L$, or $0.5S + L$, determined using importance factors equal to 1.0

4.3.2.4 Combined loads

When the total specified load is made up of loads acting for different durations, the design shall be based on the most severe combination. The load duration factor, K_D, for the load of the shortest duration may apply for that load combination, except as specified elsewhere in this Standard.

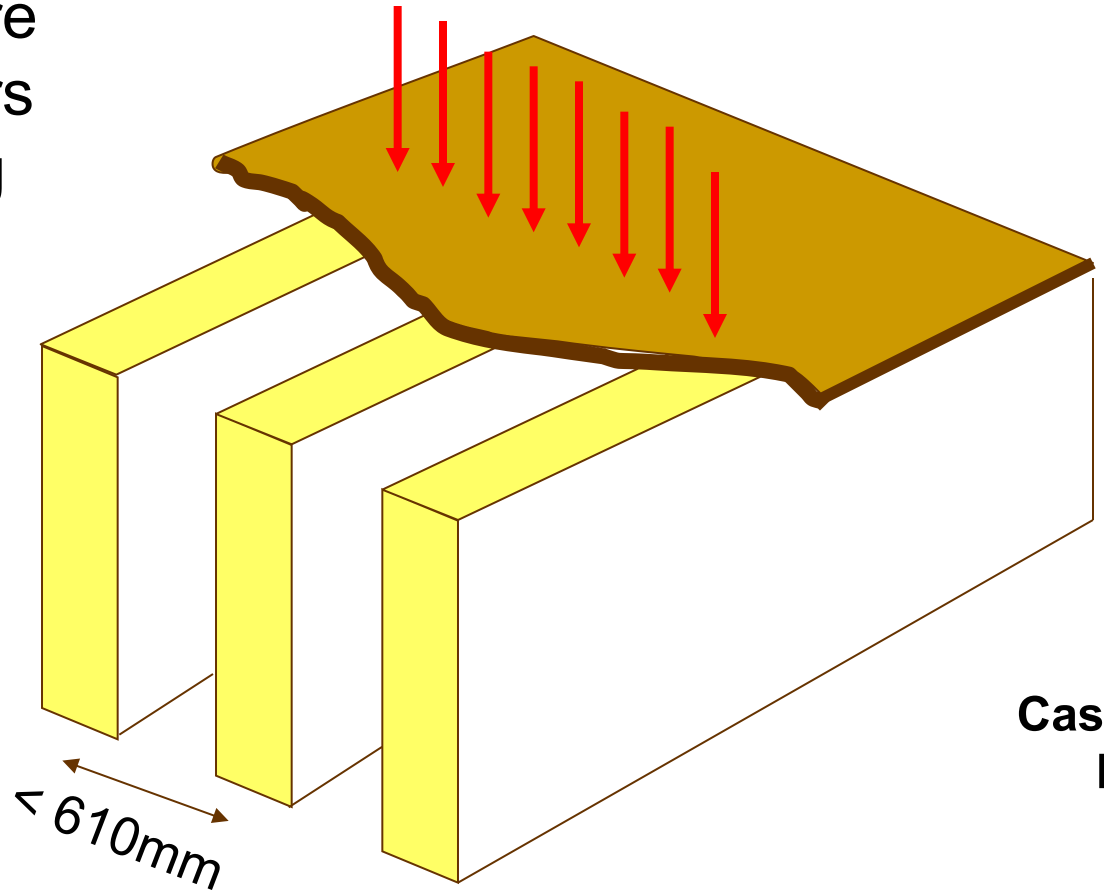
1.25D+1.4W+0.5L $K_D = 1.15$

Modification Factors

- **System Factor, K_H**
  - Repetitive members
  - Load distributed over several framing members
- Requirements:
  - Case 1: three or more parallel members spaced no more than 610 mm apart, K_H=1.1
  - Case 2: in addition, a sheathing material of a specific thickness is attached to the members, K_H=1.4

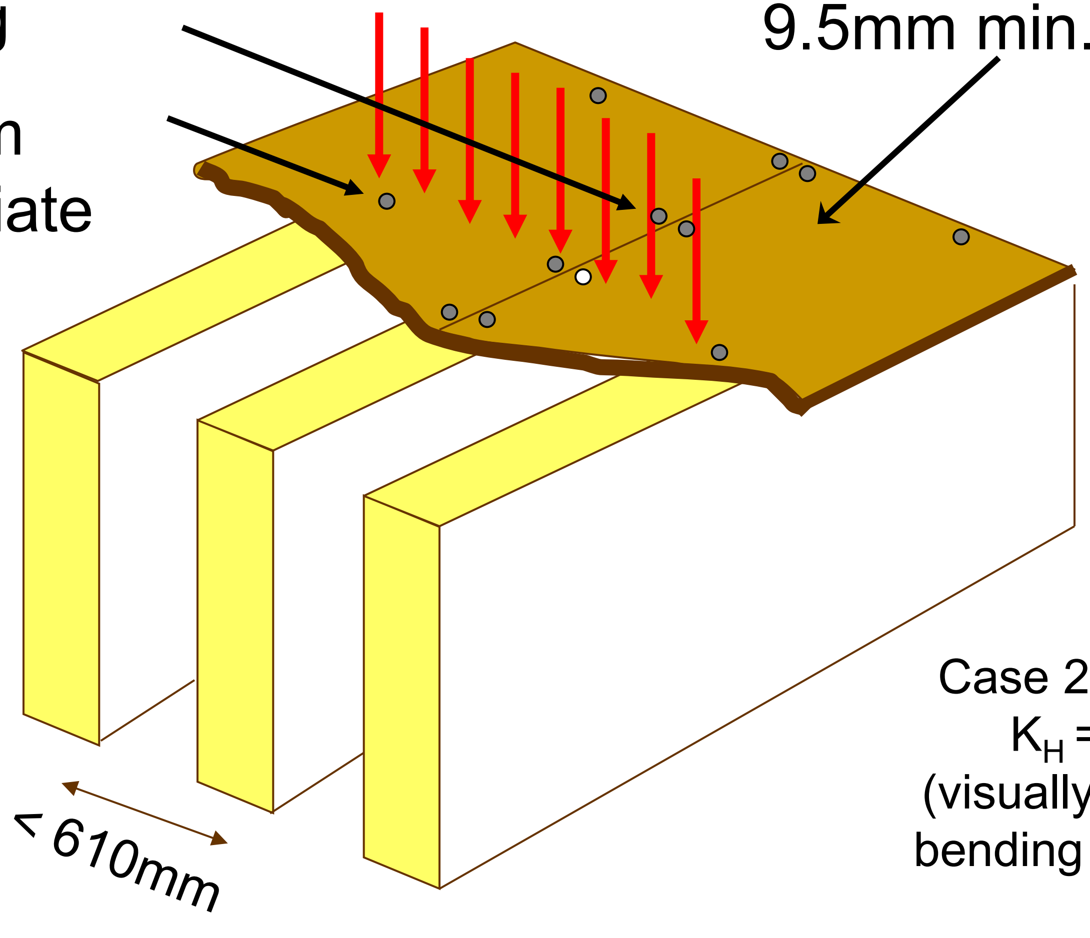
Load sharing K_H

3 or more members sharing



Case 1 system
K_H = 1.1

Load sharing K_H



Case 2 system
$K_H = 1.4$
(visually graded: bending & shear)

Modification Factors

- **System Factor, K_H**

Table 5.4.4
System factor, K_H

For specified strength in	Case 1*	Case 2† Visually graded	Case 2† MSR	Built-up beams
Bending	1.10	1.40	1.20	1.10
Longitudinal shear	1.10	1.40	1.20	1.10
Compression parallel to grain	1.10	1.10	1.10	1.00
Tension parallel to grain	1.10	—	—	1.00
All other properties	1.00	1.00	1.00	1.00

**See Clause 5.4.4.1 for conditions applying to Case 1.*
†See Clause 5.4.4.2 for conditions applying to Case 2.

Modification Factors

- **Service Condition Factor, K_S**
  - Specified strengths of wood are based on test results for dry service condition, average 15% and not exceed 19%

K_S can differ for different properties and wood products

Modification Factors

- **Service Condition Factor, K_S**

Table 5.4.2
Service condition factors, K_S

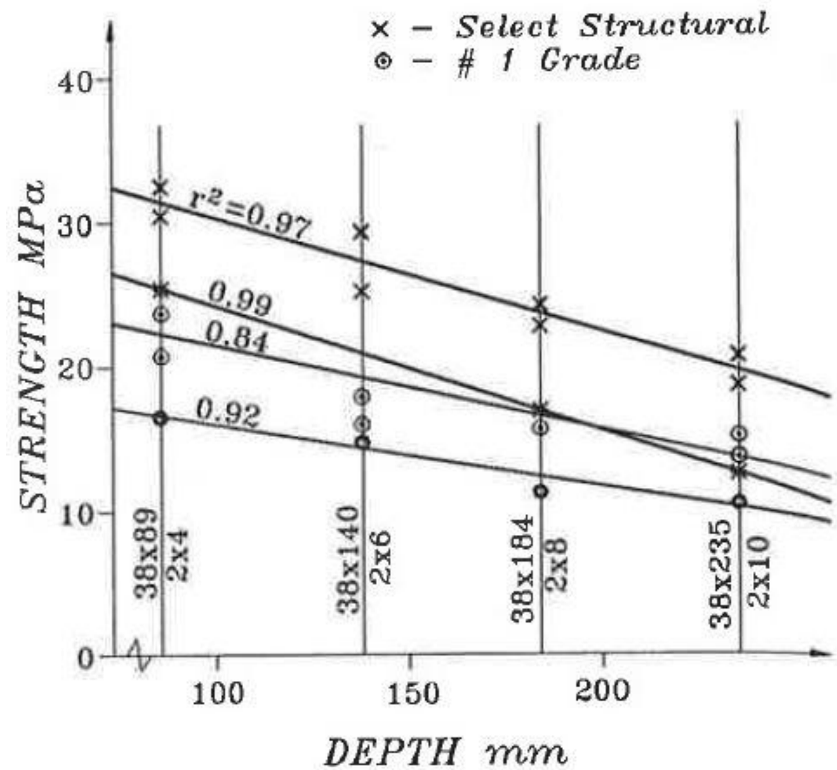
K_S	Property	Dry service conditions	Wet service conditions: sawn lumber, piling, and poles of least dimension	
			89 mm or less	Over 89 mm
K_{Sb}	Bending at extreme fibre	1.00	0.84	1.00
K_{Sv}	Longitudinal shear	1.00	0.96	1.00
K_{Sc}	Compression parallel to grain	1.00	0.69	0.91
K_{Scp}	Compression perpendicular to grain	1.00	0.67	0.67
K_{St}	Tension parallel to grain	1.00	0.84	1.00
K_{SE}	Modulus of elasticity	1.00	0.94	1.00

Modification Factors

- **Size Factor, K_z**
  - Larger members tend to be weaker than smaller members, due to the increased probability for defects

Size effect in timber

- Strength properties decrease with increasing piece size – size (or volume) effect
- Larger piece has a higher probability of containing more and larger defects



Apparent depth effect for Douglas Fir-Larch and Hem-Fir tested in bending

Modification Factors

- **Size Factor, K_z**

Table 5.4.5
Size factor, K_Z, for visually stress-graded lumber

	Bending and shear K_{Zb}, K_{Zv}			Tension parallel to grain, K_{Zt}	Compression perpendicular to grain, K_{Zcp}	Compression parallel to grain, K_{Zc}	All other properties
	Smaller dimension, mm						
Larger dimension, mm	38 to 64	89 to 102	114 or more	All	All	All	All
38	1.7	—	—	1.5	See Clause 5.5.7.5	Value computed using formula in Clause 5.5.6.2.3	1.0
64	1.7	—	—	1.5			1.0
89	1.7	1.7	—	1.5			1.0
114	1.5	1.6	1.3	1.4			1.0
140	1.4	1.5	1.3	1.3			1.0
184 to 191	1.2	1.3	1.3	1.2			1.0
235 to 241	1.1	1.2	1.2	1.1			1.0
286 to 292	1.0	1.1	1.1	1.0			1.0
337 to 343	0.9	1.0	1.0	0.9			1.0
387 or larger	0.8	0.9	0.9	0.8			1.0

Modification Factors

- **Treatment Factor, K_T**
  - Preservative treatment has little effect on structural properties
  - Incisions made for better solvent penetration may affect the strength

Incised wood for better preservative penetration

Chemical treatment of wood against decay and fire

Typically no effect, except when wood surface is incised

Fire retardant has a tendency to enhance moisture absorption leading to higher MC

Treatment Factor K_T

Table 5.4.3
Treatment factor, K_T

Product	Dry service conditions	Wet service conditions
Untreated lumber	1.00	1.00
Preservative-treated unincised lumber	1.00	1.00
Preservative-treated incised lumber of thickness 89 mm or less		
Modulus of elasticity	0.90	0.95
Other properties	0.75	0.85
Fire-retardant-treated lumber	See Clause 5.4.3.2 for effects of fire-retardant treatment.	

Investigations of the influence of incising and treating on the strength properties of larger sections did not demonstrate any consistent reductions: K_T=1.0

End

Lecture #1

Wood Handbook

Table 5–5a. Mechanical properties of some woods imported into the United States other than Canadian imports (metric)[a]—con.

Common and botanical names of species	Moisture content	Specific gravity	Static bending			Compression parallel to grain (kPa)	Shear parallel to grain (kPa)	Side hardness (N)	Sample origin[b]
			Modulus of rupture (kPa)	Modulus of elasticity (MPa)	Work to maximum load (kJ m^{-3})				
Lignumvitae (*Guaiacum* spp.)	Green	1.05	—	—	—	—	—	—	AM
	12%	—	—	—	—	78,600	—	20,000	
Limba (*Terminalia superba*)	Green	0.38	41,400	5,300	53	19,200	6,100	1,800	AF
	12%		60,700	7,000	61	32,600	9,700	2,200	
Macawood (*Platymiscium* spp.)	Green	0.94	153,800	20,800	—	72,700	12,700	14,800	AM
	12%		190,300	22,100	—	111,000	17,500	14,000	
Mahogany, African (*Khaya* spp.)	Green	0.42	51,000	7,900	49	25,700	6,400	2,800	AF
	12%		73,800	9,700	57	44,500	10,300	3,700	
Mahogany, true (*Swietenia macrophylla*)	Green	0.45	62,100	9,200	63	29,900	8,500	3,300	AM
	12%	—	79,300	10,300	52	46,700	8,500	3,600	
Manbarklak (*Eschweilera* spp.)	Green	0.87	117,900	18,600	120	50,600	11,200	10,100	AM
	12%		182,700	21,600	230	77,300	14,300	15,500	

$$P_M = P_{12}\left(\frac{P_{12}}{P_g}\right)^{\left(\frac{12-M}{M_p-12}\right)}$$